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基于共转法U.L.列式三节点壳元几何非线性有限元分析

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摘要：以三维连续介质力学和虚功原理为基础，推导出增量U.L.有限元列式，该列式保留了大位移增量刚度矩阵项，通过对该刚度矩阵进行修正可使之成为对称矩阵.根据该增量U.L.列式，文中采用三边形的形函数推导了三维三节点壳元的切线刚度矩阵，并考虑了横向剪切应力的影响.在求解增量方程时，采用CR(Co-rotational)法，将刚体位移从节点位移增量中扣除，得到节点纯变形增量，利用小应变理论计算单元内力.编制了非线性有限元程序，通过算例进行了几何非线性分析，验证该理论的精确性、高效性和通用性.

关键词：三维三节点壳元；大位移增量矩阵；U.L.列式；Co-rtational法；有限转动；非线性有限元
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Incremental Nonlinear Finite Element Analysis of 3-node Co-rotational Shell Element Based on U.L.

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Abstract：According to continuum mechanics and virtual work principle，incremental updated Lagrangian formulation (U.L.) was presented. The large displacement incremental stiffness matrix was considered in U.L.，which was rectified to be symmetrical matrix. Based on U.L.，three-dimension general-purpose three-node triangular shell element was presented when the triangular shape function was employed，transverse shear stress was also considered. During the solution of incremental equilibrium equation，the pure nodal point incremental deformation was obtained when the rigid body rotation was removed from the nodal point incremental displacement by using co-rotational procedure. Furthermore，by utilizing aforementioned theory，the nonlinear finite element program was developed. Several geometrically nonlinear numerical problems were presented to demonstrate the accuracy，effectiveness，and generality of the three dimensional three node shell element.

Key words：three-dimension three-node shell element；large displacement incremental matrix；updated Lagrangian formulation；Co-rotational approach；finite rotation theory；nonlinear finite element method

壳体结构因其承载力高、造型优美、经济实用等优点被广泛应用于工程结构设计中.壳体结构的稳定性分析，特别是屈曲分析等问题，对评估结构的安全性能具有重要意义.几何非线性有限元分析是求解该问题的有效途径[1].

根据几何刚度矩阵的计算和内力更新的方式不同，壳单元几何非线性分析可以分成以下几类：一是经典非线性壳理论[2-4]；二是三维退化非线性壳元[5-12]；三是CR法[13-25]；四是广义协调元，广义协调元主要是龙驭球等[26-28]提出的广义协调壳元.

第一种研究方式在分析中忽略横向剪切效应，在相邻单元边界上很难满足位移协调条件，难以精确模拟曲壳及中、厚壳受力特点；第二种方式可以克服第一种研究方式的缺陷，发展成通用三维壳元，但是单元推导比较复杂；第三种研究方式是最近几年发展比较快的壳单元理论，它构造简单，采用线性刚度矩阵计算几何非线性，大大缩减了计算时间，提高了求解效率；第四种研究方式本质是第一种研究方式的扩展，改善了第一种研究方式的相邻单元边界的不协调性.本文结合第二、第三种非线性壳元理论，采用Lagrangian物质描述的U.L.(updated

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Lagrangian)列式推导三维退化壳元刚度矩阵和采用 CR(Co-rotational)法扣除单元节点刚体转动得到单元节点纯变形以计算单元内力. 文献[29-31]曾对此研究方式展开理论研究与讨论.

本文从工程结构的壳体结构设计与分析的实用性、通用性、易操作性等性能出发，根据增量 U. L. 列式，推导了三维通用三节点壳元的切线刚度矩阵，并考虑了横向剪切应力的影响. 在求解增量方程过程中，采用 CR 法通过罗德里格转化矩阵(Rodrigues formula matrix)计算节点有限转动向量，再采用正交的四元数法将刚体位移从节点位移增量中扣除，得到节点纯变形增量，进而利用小应变理论计算单元内力. 根据以上理论编制了非线性有限元程序，通过几个算例的几何非线性分析，证明了该理论的精确性和通用性.

1　增量 U. L. 列式

基于连续性介质力学和虚位移原理[20,32-34]有

$$([\boldsymbol{k}_{\mathrm{L}}]+[\boldsymbol{k}_{\sigma}]+[\boldsymbol{k}_{\mathrm{U}}])\{\Delta\boldsymbol{u}_{\mathrm{e}}\}= [\boldsymbol{N}]^{\mathrm{T}}\,{}^{t+\Delta t}_{\ \ t}\{\boldsymbol{q}\}+\int_{{}^tV}[\boldsymbol{N}]^{\mathrm{T}}\,{}^{t+\Delta t}_{\ \ t}\rho^{t+\Delta t}_{\ \ t}\{\boldsymbol{f}\}\,{}^t\mathrm{d}v+ \int_{{}^tA}[\boldsymbol{N}]^{\mathrm{T}}\,{}^{t+\Delta t}_{\ \ t}\{\boldsymbol{t}\}\,{}^t\mathrm{d}a-\int_{{}^tV}[\boldsymbol{B}_{\mathrm{L}}]^{\mathrm{T}}\,{}^t_t\boldsymbol{S}\,{}^t\mathrm{d}v \tag{1}$$

式中：$[\boldsymbol{k}_{\mathrm{L}}]$为小位移刚度矩阵；$[\boldsymbol{k}_{\sigma}]$为几何刚度矩阵；$[\boldsymbol{k}_{\mathrm{U}}]$为大位移增量矩阵，$\{\Delta\boldsymbol{u}_{\mathrm{e}}\}$为节点增量位移；$[\boldsymbol{N}]$为插值函数矩阵；${}^{t+\Delta t}_{\ \ t}\{\boldsymbol{q}\}$为参考 t 时刻构形在 $t+\Delta t$时刻节点荷载向量，${}^{t+\Delta t}_{\ \ t}\rho$ 为参考 t 时刻构形在 $t+\Delta t$时刻单元体积密度；${}^{t+\Delta t}_{\ \ t}\{\boldsymbol{f}\}$为参考 t 时刻构形在 $t+\Delta t$ 时刻体荷载向量；${}^{t+\Delta t}_{\ \ t}\{\boldsymbol{t}\}$为参考 t 时刻构形在 $t+\Delta t$ 时刻面积荷载向量；${}^tV$，${}^tA$ 分别为 t 时刻构形单元的体积、面积；${}^t\mathrm{d}v$，${}^t\mathrm{d}a$ 分别为 t 时刻构形单元积分体积变量、积分面积变量；$[\boldsymbol{B}_L]$为线性应变转化刚度矩阵；${}^t_t\boldsymbol{S}$ 为 t 时刻柯西应力向量. 在基于上一时刻构形时下面公式推导过程省略左上角 $t+\Delta t$ 和左下角 t.

$$[\boldsymbol{k}_{\mathrm{L}}]=\int_{{}^tV}[\boldsymbol{B}_{\mathrm{L}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{L}}]\,{}^t\mathrm{d}v \tag{2}$$

$$[\boldsymbol{k}_{\sigma}]=\int_{{}^tV}[\boldsymbol{G}]^{\mathrm{T}}[\boldsymbol{M}][\boldsymbol{G}]\,{}^t\mathrm{d}v \tag{3}$$

$$[\boldsymbol{k}_{\mathrm{U}}]=\int_{{}^tV}(\frac{1}{2}[\boldsymbol{B}_{\mathrm{L}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{N}}]+ [\boldsymbol{B}_{\mathrm{N}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{L}}])\,{}^t\mathrm{d}v+ \int_{{}^tV}\frac{1}{2}[\boldsymbol{B}_{\mathrm{N}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{N}}]\,{}^t\mathrm{d}v \tag{4}$$

式(4)是不对称矩阵，本文进行下面简化：

$$\begin{aligned}[\boldsymbol{k}_U]=&\int_{{}^tV}(\frac{1}{2}[\boldsymbol{B}_{\mathrm{L}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{N}}]+ [\boldsymbol{B}_{\mathrm{N}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{L}}])\,{}^t\mathrm{d}v+ \int_{{}^tV}(\frac{1}{2}[\boldsymbol{B}_{\mathrm{N}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{N}}])\,{}^t\mathrm{d}v\approx\\ &\int_{{}^tV}(\frac{1}{2}[\boldsymbol{B}_{\mathrm{L}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{N}}]+ \frac{1}{2}[\boldsymbol{B}_{\mathrm{N}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{L}}])\,{}^t\mathrm{d}v+ \int_{{}^tV}(\frac{1}{2}[\boldsymbol{B}_{\mathrm{N}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{N}}])\,{}^t\mathrm{d}v=\\ &\frac{1}{2}\int_{{}^tV}([\boldsymbol{B}_{\mathrm{L}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{N}}]+ [\boldsymbol{B}_{\mathrm{N}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{L}}])\,{}^t\mathrm{d}v+ \frac{1}{2}\int_{{}^tV}([\boldsymbol{B}_{\mathrm{N}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][\boldsymbol{B}_{\mathrm{N}}])\,{}^t\mathrm{d}v\end{aligned} \tag{5}$$

式(2)～(5)中：$[\boldsymbol{D}_{\mathrm{T}}]$为增量应变对应增量应力的材料本构切线刚度矩阵；$[\boldsymbol{G}]$为几何刚度矩阵；$[\boldsymbol{M}]$为应力矩阵；$[\boldsymbol{B}_{\mathrm{N}}]$为非线性应变转化矩阵.

$[\boldsymbol{k}_{\mathrm{U}}]$矩阵对称化之后，使整个刚度矩阵对称，这对单元切线刚度矩阵元素在计算机内存中存储是有利的. Bathe[33]在推导 U. L. 列式时没有考虑大位移增量矩阵对高度非线性的影响，本文推导并修正该矩阵使其对称化.

2　三节点壳单元

2.1　单元的位移函数

如图 1 所示三节点壳单元，t 时刻单元内任意一点坐标为[5]

$${}^t\boldsymbol{x}_i=\sum_{k=1}^{N}N_k\,{}^t\boldsymbol{x}_i^k+\frac{t}{2}\sum_{k=1}^{N}a_kN_k\,{}^t\boldsymbol{V}_{ni}^k \tag{6}$$

式中：N 为节点数；N_k 为形函数，见式(7)；${}^t\boldsymbol{x}_i^k$ 为 t 时刻 k 节点位移向量；a_k 为 k 节点的壳厚度；${}^t\boldsymbol{V}_{ni}^k$ 为

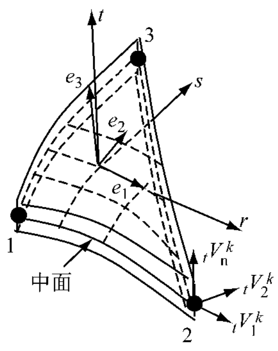


图 1　三节点壳元在 t 时刻的构形和单元随动坐标系

Fig.1　Three-node shell element in configuration at time t and CR coordinate system

节点k法向向量.

$$N_1 = r, N_2 = s, N_3 = 1 - r - s \tag{7}$$

式中:r,s,t为单元在自然坐标系的坐标分量.

壳单元上任一点t时刻位移为

$$^t\boldsymbol{u}_i = \sum_{k=1}^{N} N_k {}^t\boldsymbol{x}_i^k + \frac{t}{2}\sum_{k=1}^{N} a_k N_k ({}^t\boldsymbol{V}_{ni}^k - {}^0\boldsymbol{V}_{ni}^k) \tag{8}$$

t时刻单元任意一点的增量位移为

$$\Delta\boldsymbol{u}_i = \sum_{k=1}^{N} N_k \Delta\boldsymbol{u}_i^k + \frac{t}{2}\sum_{k=1}^{N} a_k N_k \boldsymbol{V}_{ni}^k \tag{9}$$

式中:$\boldsymbol{V}_{ni}^k$为节点k在中面法线的变化,如下式所示:

$$\boldsymbol{V}_{ni}^k = {}^{t+\Delta t}\boldsymbol{V}_{ni}^k - {}^t\boldsymbol{V}_{ni}^k$$

令α_k,β_k分别为从时刻t到时刻$t+\Delta t$向量${}^t\boldsymbol{V}_n^k$绕${}^t\boldsymbol{V}_1^k$,${}^t\boldsymbol{V}_2^k$的转角. 位移增量表达式为

$$\Delta\boldsymbol{u}_i = \sum_{k=1}^{N} N_k \Delta\boldsymbol{u}_i^k + \frac{t}{2}\sum_{k=1}^{N} a_k N_k [-{}^t\boldsymbol{V}_{2i}^k\alpha_k + {}^t\boldsymbol{V}_{1i}^k\beta_k] \tag{10}$$

对位移增量式(10)求自然坐标的偏导可表示为

$$\begin{Bmatrix} \dfrac{\partial\Delta\boldsymbol{u}_i}{\partial r} \\ \dfrac{\partial\Delta\boldsymbol{u}_i}{\partial s} \\ \dfrac{\partial\Delta\boldsymbol{u}_i}{\partial t} \end{Bmatrix} = \sum_{k=1}^{N} \begin{bmatrix} N_{k,r} & tN_{k,r}{}^t g_{1i}^k & tN_{k,r}{}^t g_{2i}^k \\ N_{k,s} & tN_{k,s}{}^t g_{1i}^k & tN_{k,s}{}^t g_{2i}^k \\ N_k & N_k{}^t g_{1i}^k & N_k{}^t g_{2i}^k \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{u}_i^k \\ \alpha_k \\ \beta_k \end{Bmatrix} \tag{11}$$

其中,

$${}^t g_{1i}^k = -\frac{1}{2}a_k{}^t\boldsymbol{V}_{2i}^k, {}^t g_{2i}^k = \frac{1}{2}a_k{}^t\boldsymbol{V}_{1i}^k \tag{12}$$

位移增量对整体坐标的偏导为

$$\begin{Bmatrix} \dfrac{\partial\Delta\boldsymbol{u}_i}{\partial^t x_1} \\ \dfrac{\partial\Delta\boldsymbol{u}_i}{\partial^t x_2} \\ \dfrac{\partial\Delta\boldsymbol{u}_i}{\partial^t x_3} \end{Bmatrix} = \sum_{k=1}^{n} \begin{bmatrix} {}_tN_{K,1} & {}^t g_{1i}^k{}_tG_1^k & {}^t g_{2i}^k{}_tG_1^k \\ {}_tN_{k,2} & {}^t g_{1i}^k{}_tG_2^k & {}^t g_{2i}^k{}_tG_2^k \\ {}_tN_{k,3} & {}^t g_{1i}^k{}_tG_3^k & {}^t g_{2i}^k{}_tG_3^k \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{u}_i^k \\ \Delta\alpha_k \\ \Delta\beta_k \end{Bmatrix} \tag{13}$$

其中,

$$\begin{aligned} {}_tN_{k,i} &= {}^t\boldsymbol{J}_{i1}^{-1}N_{k,r} + {}^t\boldsymbol{J}_{i2}^{-1}N_{k,s} \\ {}_tG_i^k &= t({}^t\boldsymbol{J}_{i1}^{-1}N_{k,r} + {}^t\boldsymbol{J}_{i2}^{-1}N_{k,s}) + {}^t\boldsymbol{J}_{i3}^{-1}N_k \end{aligned} \tag{14}$$

式中:$\boldsymbol{J}$为雅克比转化矩阵.

2.2 单元随动坐标系向量

为确定单元变形后的构形,需要建立单元的随动坐标系向量,如图1所示. 随动坐标系坐标原点取为单元的形心位置,其单位正交向量矩阵$\boldsymbol{E}$为

$$\boldsymbol{E} = (e_1 \quad e_2 \quad e_3)^{\mathrm{T}} \tag{15}$$

初始构形单元随动坐标系基向量为

$$\begin{aligned} \boldsymbol{e}_1 &= \frac{\boldsymbol{r}_{12}}{\|\boldsymbol{r}_{12}\|}, \\ \boldsymbol{e}_3 &= \frac{\boldsymbol{r}_{12}\times\boldsymbol{r}_{13}}{\|\boldsymbol{r}_{12}\times\boldsymbol{r}_{13}\|}, \\ \boldsymbol{e}_2 &= \boldsymbol{e}_3\times\boldsymbol{e}_1 \end{aligned} \tag{16}$$

式中:$\boldsymbol{r}_{ij}$为从节点i到节点j的向量.

2.3 单元小位移刚度矩阵推导

根据Green应变,从t到$t+\Delta t$时刻的位形的应变增量为

$$\Delta_t E_{ij} = \frac{1}{2}(\Delta_t u_{i,j} + \Delta_t u_{j,i} + \Delta_t u_{k,i\,t}u_{k,i}) \tag{17}$$

将式(17)改写成向量形式

$$\{\Delta_t\boldsymbol{E}_{ij}\} = \{\Delta_t\boldsymbol{E}_{ij}^{\mathrm{L}}\} + \{\Delta_t E_{ij}^{\mathrm{N}}\} \tag{18}$$

式中:$\{\Delta_t\boldsymbol{E}_{ij}^{\mathrm{L}}\}$为线性应变增量;$\{\Delta_t\boldsymbol{E}_{ij}^{\mathrm{N}}\}$为非线性应变增量. 线性应变$\{\Delta_t\boldsymbol{E}_{ij}^{\mathrm{L}}\}$可表示为

$$\begin{aligned} \{\Delta_t\boldsymbol{E}_{ij}^{\mathrm{L}}\} &= [\Delta_t E_{11}^{\mathrm{L}} \quad \Delta_t E_{22}^{\mathrm{L}} \quad 2\Delta_t E_{33}^{\mathrm{L}} \quad 2\Delta_t E_{12}^{\mathrm{L}} \\ &\quad 2\Delta_t E_{23}^{\mathrm{L}} \quad 2\Delta_t E_{31}^{\mathrm{L}}] = \\ &\quad [[{}_t^t\boldsymbol{B}_{\mathrm{L}}]^1 \quad [{}_t^t\boldsymbol{B}_{\mathrm{L}}]^2 \quad [{}_t^t\boldsymbol{B}_{\mathrm{L}}]^3]\{\Delta\boldsymbol{u}\}^{\mathrm{e}} = \\ &\quad [{}_t^t\boldsymbol{B}_{\mathrm{L}}]\{\Delta\boldsymbol{u}\}^{\mathrm{e}} \end{aligned} \tag{19}$$

$$[{}_t^t\boldsymbol{B}_{\mathrm{L}}]^k = \begin{bmatrix} {}_tN_{k,1} & 0 & 0 & {}^t g_{11}^k{}_tG_1^k & g_{21}^k{}_tG_1^k \\ 0 & {}_tN_{k,2} & 0 & {}^t g_{12}^k{}_tG_2^k & g_{22}^k{}_tG_2^k \\ 0 & 0 & {}_tN_{k,3} & {}^t g_{13}^k{}_tG_3^k & g_{23}^k{}_tG_3^k \\ {}_tN_{k,2} & {}_tN_{k,1} & 0 & B_{4,4} & B_{4,5} \\ 0 & {}_tN_{k,3} & {}_tN_{k,2} & B_{5,4} & B_{5,5} \\ {}_tN_{k,3} & 0 & {}_tN_{k,1} & B_{6,4} & B_{6,5} \end{bmatrix} \tag{20}$$

$B_{4,4} = {}^t g_{11}^k{}_tG_2^k + {}^t g_{12}^k{}_tG_1^k, B_{4,5} = {}^t g_{21}^k{}_tG_2^k + {}^t g_{22}^k{}_tG_1^k$,

$B_{5,4} = {}^t g_{12}^k{}_tG_3^k + {}^t g_{13}^k{}_tG_2^k, B_{5,5} = {}^t g_{22}^k{}_tG_3^k + {}^t g_{23}^k{}_tG_2^k$,

$B_{6,4} = {}^t g_{13}^k{}_tG_1^k + {}^t g_{11}^k{}_tG_3^k, B_{6,5} = {}^t g_{23}^k{}_tG_1^k + {}^t g_{21}^k{}_tG_3^k$

根据式(2)得到小位移刚度矩阵表达式为

$$[\boldsymbol{K}_{\mathrm{L}}] = \int_V [{}_t^t\boldsymbol{B}_{\mathrm{L}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][{}_t^t\boldsymbol{B}_{\mathrm{L}}]\mathrm{d}v \tag{21}$$

式中:$[\boldsymbol{D}_{\mathrm{T}}]$为材料的弹性本构矩阵.

$$[\boldsymbol{D}_{\mathrm{T}}] = [\boldsymbol{T}]^{\mathrm{T}}[\boldsymbol{D}][\boldsymbol{T}] \tag{22}$$

式中:$[\boldsymbol{T}]$为坐标转化矩阵[9].

2.4 单元几何刚度矩阵推导

根据式(3),三节点壳元的几何刚度矩阵为

$$[\boldsymbol{K}_\sigma] = \int_V [\boldsymbol{G}]^{\mathrm{T}}[\boldsymbol{M}][\boldsymbol{G}]\mathrm{d}v \tag{23}$$

式中:

$[\boldsymbol{G}] = [[\boldsymbol{G}]^1 \quad [\boldsymbol{G}]^2 \quad [\boldsymbol{G}]^3]$

$$[\boldsymbol{G}]^k=\begin{bmatrix} {}_tN_{k,1} & 0 & 0 & {}^tg_{11}^k\,{}_tG_1^k & {}^tg_{21}^k\,{}_tG_1^k \\ {}_tN_{k,2} & 0 & 0 & {}^tg_{11}^k\,{}_tG_2^k & {}^tg_{21}^k\,{}_tG_2^k \\ {}_tN_{k,3} & 0 & 0 & {}^tg_{11}^k\,{}_tG_3^k & {}^tg_{21}^k\,{}_tG_3^k \\ 0 & {}_tN_{k,1} & 0 & {}^tg_{12}^k\,{}_tG_1^k & {}^tg_{22}^k\,{}_tG_1^k \\ 0 & {}_tN_{k,2} & 0 & {}^tg_{12}^k\,{}_tG_2^k & {}^tg_{22}^k\,{}_tG_2^k \\ 0 & {}_tN_{k,3} & 0 & {}^tg_{12}^k\,{}_tG_3^k & {}^tg_{22}^k\,{}_tG_3^k \\ 0 & 0 & {}_tN_{k,1} & {}^tg_{13}^k\,{}_tG_1^k & {}^tg_{23}^k\,{}_tG_1^k \\ 0 & 0 & {}_tN_{k,2} & {}^tg_{13}^k\,{}_tG_2^k & {}^tg_{23}^k\,{}_tG_2^k \\ 0 & 0 & {}_tN_{k,3} & {}^tg_{13}^k\,{}_tG_3^k & {}^tg_{23}^k\,{}_tG_3^k \end{bmatrix}$$

$$[\boldsymbol{M}]=\begin{bmatrix} [\boldsymbol{\sigma}]_{3\times3} & \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & [\boldsymbol{\sigma}]_{3\times3} & \boldsymbol{0}_{3\times3} \\ \boldsymbol{0}_{3\times3} & \boldsymbol{0}_{3\times3} & [\boldsymbol{\sigma}]_{3\times3} \end{bmatrix}$$

$$[\boldsymbol{\sigma}]_{3\times3}=\begin{bmatrix} {}_t^t\sigma_{11} & {}_t^t\sigma_{12} & {}_t^t\sigma_{13} \\ {}_t^t\sigma_{21} & {}_t^t\sigma_{22} & {}_t^t\sigma_{23} \\ {}_t^t\sigma_{31} & {}_t^t\sigma_{32} & {}_t^t\sigma_{33} \end{bmatrix}$$

2.5　单元大位移矩阵推导

式(18)中的非线性应变增量为

$$\Delta_t\boldsymbol{E}_{ij}^{\mathrm{N}}=[\Delta_tE_{11}^{\mathrm{N}}\quad \Delta_tE_{22}^{\mathrm{N}}\quad 2\Delta_tE_{33}^{\mathrm{N}}\quad 2\Delta_tE_{12}^{\mathrm{N}}\quad 2\Delta_tE_{23}^{\mathrm{N}}\quad 2\Delta_tE_{31}^{\mathrm{N}}]=\frac{1}{2}[{}_t\boldsymbol{\Theta}]\{{}_t\boldsymbol{\theta}\} \tag{24}$$

$$[{}_t\boldsymbol{\Theta}]=\begin{bmatrix} \{\boldsymbol{\theta}_1\}^{\mathrm{T}} & \{\boldsymbol{0}\} & \{\boldsymbol{0}\} \\ \{\boldsymbol{0}\} & \{\boldsymbol{\theta}_2\}^{\mathrm{T}} & \{\boldsymbol{0}\} \\ \{\boldsymbol{0}\} & \{\boldsymbol{0}\} & \{\boldsymbol{\theta}_3\}^{\mathrm{T}} \\ \{\boldsymbol{\theta}_2\}^{\mathrm{T}} & \{\boldsymbol{\theta}_1\}^{\mathrm{T}} & \{\boldsymbol{0}\} \\ \{\boldsymbol{0}\} & \{\boldsymbol{\theta}_3\}^{\mathrm{T}} & \{\boldsymbol{\theta}_2\}^{\mathrm{T}} \\ \{\boldsymbol{\theta}_3\}^{\mathrm{T}} & \{\boldsymbol{0}\} & \{\boldsymbol{\theta}_1\}^{\mathrm{T}} \end{bmatrix} \tag{25}$$

$$\{{}_t\boldsymbol{\theta}\}=[\{\boldsymbol{\theta}_1\}^{\mathrm{T}}\quad \{\boldsymbol{\theta}_2\}^{\mathrm{T}}\quad \{\boldsymbol{\theta}_3\}^{\mathrm{T}}]^{\mathrm{T}} \tag{26}$$

$$\{\boldsymbol{\theta}_i\}=\left[\frac{\partial\Delta u_1}{\partial^t x_i}\quad \frac{\partial\Delta u_2}{\partial^t x_i}\quad \frac{\partial\Delta u_3}{\partial^t x_i}\right]^{\mathrm{T}} \tag{27}$$

式(26)可以表示为

$$\{{}_t\boldsymbol{\theta}\}=[\boldsymbol{G}]\{\Delta\boldsymbol{u}\}^{\mathrm{e}} \tag{28}$$

将式(28)带入式(24)并变分得到

$$[{}_t^t\boldsymbol{B}_{\mathrm{N}}]=[\boldsymbol{\Theta}][\boldsymbol{G}] \tag{29}$$

根据式(5)就可以得到三节点壳元的大位移刚度矩阵.

$$[\boldsymbol{k}_U]=\frac{1}{2}\int_{{}^tV}([{}_t^t\boldsymbol{B}_{\mathrm{L}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][{}_t^t\boldsymbol{B}_{\mathrm{N}}]+[{}_t^t\boldsymbol{B}_{\mathrm{N}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][{}_t^t\boldsymbol{B}_{\mathrm{L}}])^t\mathrm{d}v+\frac{1}{2}\int_{{}^tV}([{}_t^t\boldsymbol{B}_{\mathrm{N}}]^{\mathrm{T}}[\boldsymbol{D}_{\mathrm{T}}][{}_t^t\boldsymbol{B}_{\mathrm{N}}])^t\mathrm{d}v \tag{30}$$

在增量迭代步求解过程中，利用每次迭代步中上步收敛解的增量位移去求解大位移增量矩阵.

3　单元状态更新

单元节点 k 从时刻 t 到时刻 $t+\Delta t$ 需要坐标更新.

$${}^{t+\Delta t}\boldsymbol{x}_k={}^t\boldsymbol{x}_k+\Delta\boldsymbol{u}_k \tag{31}$$

式中：${}^{t+\Delta t}\boldsymbol{x}_k$ 为 k 节点在时刻 $t+\Delta t$ 的坐标值；${}^t\boldsymbol{x}_k$ 为 k 节点在时刻 t 的坐标值；$\Delta\boldsymbol{u}_k$ 为 k 节点在全局坐标系下的增量位移解.

此时单元随动坐标系的单位正交基向量需按照式(16)进行更新.

3.1　节点转动向量状态更新

从时刻 t 到时刻 $t+\Delta t$ 单元节点 k 的罗德里格转化矩阵为[15-16]

$${}_t^{t+\Delta t}\boldsymbol{R}_k=\boldsymbol{I}+\frac{1}{\frac{1}{4}\Delta^{t+\Delta t}_{\ t}\boldsymbol{\theta}_k^{\mathrm{T}}\,{}^{t+\Delta t}_{\ t}\boldsymbol{\theta}_k}\cdot\left[\Delta^{t+\Delta t}_{\ t}\boldsymbol{\Theta}+\frac{1}{2}(\Delta^{t+\Delta t}_{\ t}\boldsymbol{\Theta}_k)^2\right] \tag{32}$$

式中：$\boldsymbol{I}$ 为 3×3 单位矩阵；$\Delta^{t+\Delta t}_{\ t}\boldsymbol{\Theta}_k$ 见文献[16]，则节点坐标系单位正交基矢量进行如下更新：

$${}^{t+\Delta t}_{\ t}\boldsymbol{V}_k=({}^{t+\Delta t}_{\ t}\boldsymbol{R}_k){}_t^t\boldsymbol{V}_k \tag{33}$$

3.2　节点纯变形计算

单元节点 k 在随动坐标系的平动变形为

$$\Delta^{t+\Delta t}_{\ t}\boldsymbol{u}^k=({}^{t+\Delta t}_{\ t}\boldsymbol{E}^k)^{\mathrm{T}}({}^{t+\Delta t}_{\ t}\boldsymbol{x})-({}_t\boldsymbol{E}^k)^{\mathrm{T}}\,{}_t\boldsymbol{x} \tag{34}$$

节点转动变形向量采用正交四元数 $\boldsymbol{q}$ 向量进行计算.

$$\boldsymbol{q}=\{q_0\quad q_1\quad q_2\quad q_3\}^{\mathrm{T}} \tag{35}$$

本文采用 Crisfield 算法[35]，具体如下：

if　$q_0<0$

$\qquad\boldsymbol{q}=-\boldsymbol{q}$

$\qquad$if　$\|\boldsymbol{q}\|==0\quad\boldsymbol{\theta}=0$

$\qquad$else if　$\|\boldsymbol{q}\|<q_0$

$$\boldsymbol{\theta}=2\,\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|}\arcsin\|\boldsymbol{q}\|$$

else　$\boldsymbol{\theta}=2\,\dfrac{\boldsymbol{q}}{\|\boldsymbol{q}\|}\arccos q_0$ (36)

结合式(34)，(36)得到单元节点在随动坐标系的节点纯变形向量为

$$\Delta\tilde{\boldsymbol{u}}=(\Delta\tilde{u}_1\quad \Delta\tilde{u}_2\quad \Delta\tilde{u}_3\quad \Delta\tilde{\theta}_1\quad \Delta\tilde{\theta}_2\quad \Delta\tilde{\theta}_3)^{\mathrm{T}} \tag{37}$$

3.3　单元内力计算

$t+\Delta t$ 时刻单元节点的变形很小，可采用小应变理论[30]，根据材料弹性刚度矩阵和线性应变矩阵计

算当前时刻的 Cauchy 应力.

$$\{{}^{t+\Delta t}_{t}\boldsymbol{\sigma}\} = \{{}^{t}\boldsymbol{\sigma}\} + [\boldsymbol{D}][{}_{t}\boldsymbol{B}]\{\Delta\widetilde{\boldsymbol{u}}\} \tag{38}$$

式中:$[\boldsymbol{D}]$为材料弹性刚度矩阵,若考材料弹塑性,可以将$[\boldsymbol{D}]$改为材料弹塑性刚度矩阵$[\boldsymbol{D}_{ep}]$即可考虑材料弹塑性.

则 t 时刻单元的节点内力在整体坐标下计算如下,

$$\{{}_{t}\boldsymbol{f}\} = [{}^{t}\boldsymbol{R}]^{\mathrm{T}}\int_{{}^{t}V}[{}^{t}_{t}\boldsymbol{B}]^{\mathrm{T}}\{{}^{t}\boldsymbol{\sigma}\}{}^{t}\mathrm{d}v \tag{39}$$

式中:$[{}^{t}\boldsymbol{R}]^{\mathrm{T}}$ 见文献[36].

结合式(21),(23),(30),(39)得到 U. L. 列式的大位移增量方程为

$$([{}^{t+\Delta t}_{t}\boldsymbol{K}_{\mathrm{L}}] + [{}^{t+\Delta t}_{t}\boldsymbol{K}_{\sigma}] + [{}^{t+\Delta t}_{t}\boldsymbol{K}_{\mathrm{U}}])\{\Delta\boldsymbol{U}\} = \{{}^{t+\Delta t}_{t}\boldsymbol{F}\} - \{{}_{t}\boldsymbol{f}\} \tag{40}$$

式中:$[{}^{t+\Delta t}_{t}\boldsymbol{K}_{\mathrm{L}}]$为整体坐标系下小位移刚度矩阵;$[{}^{t+\Delta t}_{t}\boldsymbol{K}_{\sigma}]$为整体坐标系下的几何刚度矩阵;$[{}^{t+\Delta t}_{t}\boldsymbol{K}_{\mathrm{U}}]$为整体坐标系下的大位移刚度矩阵;$\{{}^{t+\Delta t}_{t}\boldsymbol{F}\}$为结构在 $t+\Delta t$ 时刻所受的外荷载向量;$\{{}_{t}\boldsymbol{f}\}$为结构在 t 时刻内力向量.

本文在求解式(40)的增量位移时采用修正的弧长法[36-37].

4　算例分析

4.1　悬臂梁大转动分析

图 2 所示为自由端作用集中弯矩的悬臂梁. 已知 $L=8$ m, 划分为 8 等分,截面为 1.0 m×0.1 m,弹性模量为 210 GPa,泊松比为 0.2,剪切模量为 105 GPa,施加悬臂梁自由端的弯矩为 $M=k\pi EI/L$,其中 k 值分别取 0.6,0.5,0.4,0.3,0.2,0.1. 变形结果如图 3 所示. 由于集中弯矩是常数,变形后的悬臂梁形状趋近于一个完美的圆. $k=0.3$ 和 0.6 时悬臂梁的变形值见表 1,所对应的结构变形图见图 4. 计算结果表明,该壳元可精确模拟中等大转动问题.

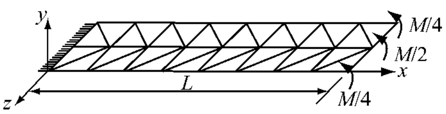


图 2　自由端作用集中弯矩的悬臂梁

Fig.2　Cantilever beam subjected to concentrate moment

4.2　柱壳大位移分析

图 5 所示为受集中荷载作用的铰接柱壳. 已知 $R=2\ 540$ mm, $L=254$ mm, $t=12.7$ mm(6.35 mm), $\theta=0.1$ rad,弹性模量$E=3\ 102.75$ MPa,泊

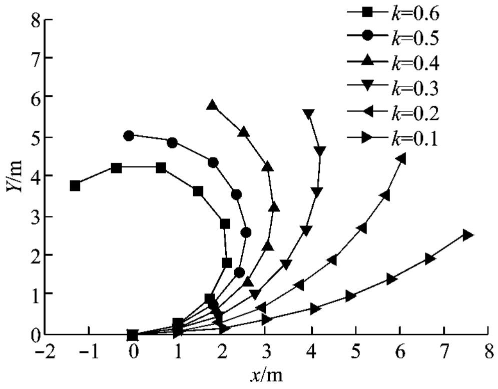


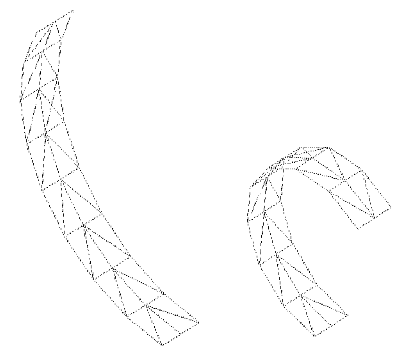
图 3　对应不同 k 值的悬臂梁变形图

Fig.3　Configuration of the deformed cantilever beam according to different k

表 1　$k=0.3$ 和 $k=0.6$ 时悬臂梁的变形

Tab.1　Deflection of cantilever beam according to $k=0.3$ and $k=0.6$

自原点长度/m	x/m		y/m	
	k=0.3	k=0.6	k=0.3	k=0.6
1.0	0.993	0.972	0.118	0.234
2.0	1.931	1.732	0.467	0.886
3.0	2.761	2.112	1.027	1.814
4.0	3.437	2.027	1.767	2.816
5.0	3.919	1.490	2.648	3.671
6.0	4.190	0.626	3.618	4.203
7.0	4.193	−0.390	4.627	4.220
8.0	3.960	−1.315	5.612	3.770



a k=0.3　　b k=0.6

图 4　对应 $k=0.3$, $k=0.6$ 的结构变形图

Fig.4　Deformed structures responding to $k=0.5$, $k=0.6$

松比 $\nu=0.3$,沿直角边铰接,曲边自由. 此算例被许多知名学者研究过,例如 Gal 和 Levy[21],Surana[10],Crisfield[35].

对图 5 所示柱壳采用 8×8 网格剖分,分别对图中 A,B 点进行 $t=12.7$ mm 和 $t=6.35$ mm 的几何非线性大位移计算. 在计算 $t=12.7$ mm 时,柱壳采用 25 步荷载增量,每个增量步最多只需 3 个迭代步便可以满足能量误差(残余力向量乘以增量位移)为

1×10^{-6}的收敛条件；在计算 $t=6.35$ mm 时，柱壳采用 35 步荷载增量，每个增量步也最多只需 3 个迭代步便可以满足能量误差为 1×10^{-6}的收敛条件，其荷载-挠度曲线见图 6，图 7. 本文计算结果与 Surana[10] 和 Gal[21]结果相当吻合.

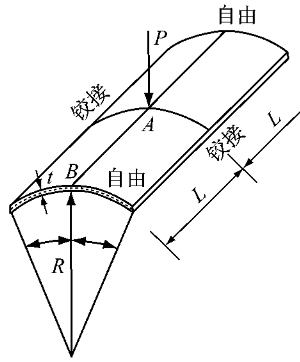


图 5　集中荷载作用的圆柱壳

Fig.5　Hinged cylindrical shell subjected to point load

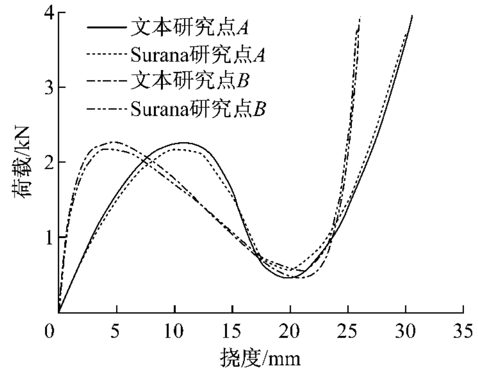


图 6　集中荷载作用的圆柱壳荷载-挠度曲线($t=12.7$ mm)

Fig. 6　Load-deflection curves for hinged cylindrical shell subjected to point load($t=12.7$ mm)

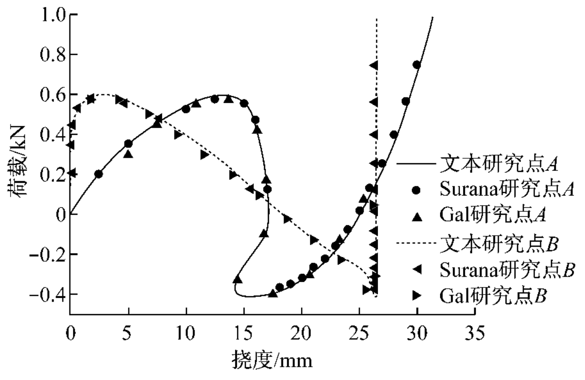


图 7　集中荷载作用的圆柱壳荷载-挠度曲线($t=6.35$ mm)

Fig. 7　Load-deflection curves for hinged cylindrical shell subjected to point load($t=6.35$ mm)

如图 7 所示，当厚度 $t=6.35$ mm 时，分岔点和极值点(最小极值点)在结构后屈曲阶段开始分离，且分岔点提前发生于极值点，这种情况对薄壳大变形稳定是不利的，而 $t=12.7$ mm 时没有此情况发生，由此证明该壳元可以对中厚壳、薄壳几何非线性大位移进行前屈曲分析和后屈曲分析.

5　结论

从壳体结构设计与分析的实用性出发，根据增量 U. L. 列式，推导了三维通用壳元. 在求解增量方程时，采用 CR 法通过罗德里格转化矩阵(Rodrigues formula matrix)计算节点有限转动向量，再采用正交的四元数法将刚体位移从节点位移增量中扣除，得到节点纯变形增量，进而利用小应变理论计算单元内力. 编制了非线性有限元程序，通过算例的几何非线性分析，可以得出以下结论：

(1) 利用 U. L. 列式推导单元刚度矩阵并采用 CR 法进行单元应力、内力更新，使每增量步计算中的迭代次数大大降低，一般只需 2～3 次迭代就可以满足收敛条件，加快了收敛速度，提高了计算效率.

(2) 推导的三节点三维壳元可以精确模拟大位移、大转动问题.

(3) 利用推导的三节点三维壳元进行中厚壳、薄壳大位移几何非线性屈曲分析是可靠有效的.

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