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Dynamic behavior of commensal symbiosis system with both feedback control and Allee effect

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Abstract: The dynamic behavior of commensal symbiosis system with both feedback control and Allee effect is complex. In this paper, the influence of additive Allee effect on the dynamical behavior of the system has been studied based on the feedback control commensal symbiosis system. It is proved that the conditions for the existence of positive equilibrium point are sufficient to ensure the global attractive of the system. By constructing an appropriate Lyapunov function, the sufficient conditions for the global stability of the equilibrium point are obtained. The research shows that Allee effect does not affect the stability of the equilibrium point of the feedback control biased commensal symbiosis system, but affects the position of the equilibrium point.

Key words: commensal symbiosis system; feedback control; Allee effect; global stability

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具反馈控制和Allee效应的偏利共生系统动力学行为

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摘　要：具有反馈控制和Allee效应的偏利共生模型，其动力学行为比较复杂.在反馈控制偏利共生模型的基础上，研究加法Allee效应对系统的动力学行为的影响，证明了系统正平衡点存在的条件足以保证系统的全局吸引性.通过构造适当的Lyapunov函数，得到平衡点全局稳定的充分条件.研究表明：Allee效应不影响反馈控制偏利共生系统平衡点的稳定性，但会影响平衡点的位置.

关键词：偏利共生系统；反馈控制；Allee效应；全局稳定性

1 Introduction

Human interference with biological resources forms a population model with feedback control. Since

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GOPALSAMY et al [1] proposed the autonomous single species feedback control model, many scholars have carried out research on the feedback control model. HAN et al[2] studied the commensal symbiosis model with feedback control, but the author did not consider the influence of Allee effect on the population.

Allee effect, which exists in many endangered species. The research on dynamic behavior of biological population model with Allee effect has also become a new hotspot[3-6]. Most of the studies consider both feedback control and Allee effect are only in the single population model[7-9], and they have mainly aimed at the predator-prey system, while there are few studies on the biased symbiotic system considering Allee effect.

Therefore, on the basis of [2], this paper adds the additive Allee effect, and proposes the following biased commensal symbiosis model with feedback control and Allee effect:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t}=x\left(b_1-a_{11}x-\dfrac{\alpha}{x+\gamma}\right)+a_{12}xy-c_1x\mu, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t}=y(b_2-a_{22}y)-c_2yv, \\ \dfrac{\mathrm{d}\mu}{\mathrm{d}t}=-p_1\mu+q_1x, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t}=-p_2v+q_2y, \end{cases} \tag{1}$$

where, b_i, c_i, p_i, q_i, $a_{ij}(i,\ j=1,2)$ are all positive constants, x, y represent the densities of the two populations at time t, respectively, μ, v is a feedback control variable, and $\dfrac{\alpha}{x+\gamma}$ indicates Allee effect. The global attractive of the system and stability of positive equilibrium under certain conditions are discussed below.

2 Positive equilibrium

It is easy to know by calculation, if condition $\alpha<b_1\gamma$ holds. System (1) has a unique positive equilibrium point $E^*(x^*, y^*, \mu^*, v^*)$, where

$$x^*=\frac{-B+\sqrt{\Delta}}{2A},\ y^*=\frac{b_2p_2}{a_{22}p_2+c_2q_2},\ \mu^*=\frac{q_1}{p_1}x^*,\ v^*=\frac{q_2}{p_2}y^*,\ A=\frac{a_{11}p_1+c_1q_1}{p_1},$$

$$B=\frac{\gamma(a_{11}p_1+c_1q_1)}{p_1}-\frac{b_1(a_{22}p_2+c_2q_2)+a_{12}b_2p_2}{a_{22}p_2+c_2q_2},\ C=-\frac{(b_1\gamma-\alpha)(a_{22}p_2+c_1q_2)+\gamma a_{12}b_2p_2}{a_{22}p_2+c_1q_2}.$$

3 Global Attractive

Theorem 1 Assume that $\alpha<b_1\gamma$, then the unique positive equilibrium point $E^*(x^*, y^*, \mu^*, v^*)$ of the system (1) is globally attractive, i.e.,

$$\lim_{t\to+\infty}x(t)=x^*,\ \lim_{t\to+\infty}y(t)=y^*,\ \lim_{t\to+\infty}\mu(t)=\mu^*,\ \lim_{t\to+\infty}v(t)=v^*. \tag{2}$$

Proof Let us suppose that

$$U_1=\limsup_{t\to+\infty}x(t),\ V_1=\liminf_{t\to+\infty}x(t),\ U_2=\limsup_{t\to+\infty}y(t),\ V_2=\liminf_{t\to+\infty}y(t),$$

$$U_3=\limsup_{t\to+\infty}\mu(t),\ V_3=\liminf_{t\to+\infty}\mu(t),\ U_4=\limsup_{t\to+\infty}v(t),\ V_4=\liminf_{t\to+\infty}v(t).$$

For the convenience of expression, we let the first, second, third and fourth equations of system (1) be equations (a), (b), (c) and (d) respectively.

From the equations of systems(1)(b), it follows that

$$\frac{\mathrm{d}y}{\mathrm{d}t}\leqslant y(b_2-a_{22}y). \tag{3}$$

According to lemma 1.1.4[10] and the comparison principle of differential equations, there is

$$U_2=\limsup_{t\to+\infty} y(t)\leqslant\frac{b_2}{a_{22}}\triangleq M_2^{(1)}.\tag{4}$$

Hence, for small enough $\varepsilon>0$, it is known from formula (4) that, there exists big enough $T_1>0$, for all $t>T_1$, we have

$$y(t)<M_2^{(1)}+\varepsilon.\tag{5}$$

By (5), combined with (a), we have

$$\frac{\mathrm{d}x}{\mathrm{d}t}\leqslant x(b_1+a_{12}(M_2^{(1)}+\varepsilon)-a_{11}x).\tag{6}$$

Then,

$$U_1=\limsup_{t\to+\infty} x(t)\leqslant\frac{b_1+a_{12}(M_2^{(1)}+\varepsilon)}{a_{11}}.\tag{7}$$

Letting $\varepsilon\to 0$, we have

$$U_1=\limsup_{t\to+\infty} x(t)\leqslant\frac{b_1+a_{12}M_2^{(1)}}{a_{11}}\triangleq M_1^{(1)}.\tag{8}$$

Similarly, from (5) and (d), we have

$$\frac{\mathrm{d}v}{\mathrm{d}t}\leqslant -p_2v+q_2(M_2^{(1)}+\varepsilon).\tag{9}$$

Then,

$$U_4=\limsup_{t\to+\infty} v(t)\leqslant\frac{q_2M_2^{(1)}}{p_2}\triangleq M_4^{(1)}.\tag{10}$$

For small enough $\varepsilon>0$, it is known from (4) that, there exists big enough $T_2>T_1$, for all $t>T_2$, we have

$$x(t)<M_1^{(1)}+\varepsilon.\tag{11}$$

From (10) and (c), we have

$$\frac{\mathrm{d}\mu}{\mathrm{d}t}\leqslant -p_1\mu+q_1(M_1^{(1)}+\varepsilon).\tag{12}$$

Therefore,

$$U_3=\limsup_{t\to+\infty} \mu(t)\leqslant\frac{q_1M_1^{(1)}}{p_1}\triangleq M_3^{(1)}.\tag{13}$$

Similarly, we have

$$\frac{\mathrm{d}y}{\mathrm{d}t}\geqslant y(b_2-a_{22}y)-c_2y(M_4^{(1)}+\varepsilon),\tag{14}$$

hence

$$V_2=\liminf_{t\to+\infty} y(t)\geqslant\frac{b_2-c_2M_4^{(1)}}{a_{22}}\triangleq m_2^{(1)}.\tag{15}$$

Then,

$$\frac{\mathrm{d}x}{\mathrm{d}t}\geqslant x\left(b_1-a_{11}x-\frac{\alpha}{x+\gamma}\right)-c_1(M_3^{(1)}+\varepsilon)x\geqslant x\left(b_1-a_{11}x-\frac{\alpha}{\gamma}\right)-c_1(M_3^{(1)}+\varepsilon)x.\tag{16}$$

Therefore,

$$V_1=\liminf_{t\to+\infty} x(t)\geqslant\frac{b_1-\frac{\alpha}{\gamma}-c_1M_3^{(1)}}{a_{11}}\triangleq m_1^{(1)}\ .\tag{17}$$

Applying the same method, we have

$$\frac{\mathrm{d}v}{\mathrm{d}t} \geqslant -p_2 v + q_2(m_2^{(1)} + \varepsilon) \Rightarrow V_4 = \liminf_{t\to+\infty} v(t) \geqslant \frac{q_2 m_2^{(1)}}{p_2} \triangleq m_4^{(1)}, \tag{18}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} \geqslant -p_1 \mu + q_1(m_1^{(1)} + \varepsilon) \Rightarrow V_3 = \liminf_{t\to+\infty} \mu(t) \geqslant \frac{q_1 m_1^{(1)}}{p_1} \triangleq m_3^{(1)}, \tag{19}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} \leqslant y(b_2 - a_{22} y) - c_2 y(m_4^{(1)} + \varepsilon) \Rightarrow U_2 \leqslant \frac{b_2 - c_2 m_4^{(1)}}{a_{22}} \triangleq M_2^{(2)}, \tag{20}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} \leqslant x\left(b_1 - a_{11}x - \frac{\alpha}{M_1^{(1)} + \varepsilon + \gamma}\right) + a_{12}x(M_2^{(2)} + \varepsilon) - c_1 x(m_3^{(1)} + \varepsilon) \Rightarrow U_1 \leqslant \frac{b_1 + a_{12}M_2^{(2)} - c_1 m_3^{(1)} - \dfrac{\alpha}{M_1^{(1)} + \gamma}}{a_{11}} \triangleq M_1^{(2)}, \tag{21}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} \leqslant -p_2 v + q_2(M_2^{(2)} + \varepsilon) \Rightarrow U_4 \leqslant \frac{q_2 M_2^{(2)}}{p_2} \triangleq M_4^{(2)}, \tag{22}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} \leqslant -p_1 \mu + q_1(M_1^{(2)} + \varepsilon) \Rightarrow U_3 \leqslant \frac{q_1 M_1^{(2)}}{p_1} \triangleq M_3^{(2)}, \tag{23}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} \geqslant y(b_2 - a_{22} y) - c_2 y(M_4^{(2)} + \varepsilon) \Rightarrow V_2 \geqslant \frac{b_2 - c_2 M_4^{(2)}}{a_{22}} \triangleq m_2^{(2)}, \tag{24}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} \geqslant x\left(b_1 - a_{11}x - \frac{\alpha}{m_1^{(1)} + \varepsilon + \gamma}\right) + a_{12}x(m_2^{(2)} + \varepsilon) - c_1(M_3^{(2)} + \varepsilon)x \Rightarrow V_1 \geqslant \frac{b_1 + a_{12}m_2^{(2)} - c_1 M_3^{(2)} - \dfrac{\alpha}{m_1^{(1)} + \gamma}}{a_{11}} \triangleq m_1^{(2)}, \tag{25}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} \geqslant -p_2 v + q_2(m_2^{(2)} + \varepsilon) \Rightarrow V_4 \geqslant \frac{q_2 m_2^{(2)}}{p_2} \triangleq m_4^{(2)}, \tag{26}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} \geqslant -p_1 \mu + q_1(m_1^{(2)} + \varepsilon) \Rightarrow V_3 \geqslant \frac{q_1 m_1^{(2)}}{p_1} \triangleq m_3^{(2)}. \tag{27}$$

Repeating the above process, we get eight sequences $\{M_i^{(n)}\}$, $\{m_i^{(n)}\}$ $(i=1, 2, 3, 4)$, with

$$M_1^{(n)} = \frac{b_1 + a_{12}M_2^{(n)} - c_1 m_3^{(n-1)} - \dfrac{\alpha}{M_1^{(n-1)} + \gamma}}{a_{11}},\ M_2^{(n)} = \frac{b_2 - c_2 m_4^{(n-1)}}{a_{22}},\ M_3^{(n)} = \frac{q_1 M_1^{(n)}}{p_1},\ M_4^{(n)} = \frac{q_2 M_2^{(n)}}{p_2},$$

$$m_1^{(n)} = \frac{b_1 + a_{12}m_2^{(n)} - c_1 M_3^{(n-1)} - \dfrac{\alpha}{m_1^{(n-1)} + \gamma}}{a_{11}},\ m_2^{(n)} = \frac{b_2 - c_2 M_4^{(n)}}{a_{22}},\ m_3^{(n)} = \frac{q_1 m_1^{(n)}}{p_1},\ m_4^{(n)} = \frac{q_2 m_2^{(n)}}{p_2}.$$

Obviously, for any integer n, when $t \geqslant T_{2n}$, we have $m_i^{(n)} \leqslant V_i \leqslant U_i \leqslant M_i^{(n)}$ $(i=1, 2, 3, 4)$. By mathematical induction, we can prove that $\{M_i^{(n)}\}$ is strictly monotonically reduced, and $\{m_i^{(n)}\}$ is strictly monotonically increased. Thus, both $\lim\limits_{n\to\infty} M_i^{(n)}$ and $\lim\limits_{n\to\infty} m_i^{(n)}$ $(i=1, 2, 3, 4)$ exist.

Assume that $\lim\limits_{n\to\infty} M_1^{(n)} = \overline{x}$, $\lim\limits_{n\to\infty} M_2^{(n)} = \overline{y}$, $\lim\limits_{n\to\infty} M_3^{(n)} = \overline{u}$, $\lim\limits_{n\to\infty} M_4^{(n)} = \overline{v}$, and $\lim\limits_{n\to\infty} m_1^{(n)} = \underline{x}$, $\lim\limits_{n\to\infty} m_2^{(n)} = \underline{y}$, $\lim\limits_{n\to\infty} m_3^{(n)} = \underline{u}$, $\lim\limits_{n\to\infty} m_4^{(n)} = \underline{v}$. Substitute these quantities into system (1), through calculation, we have, $\overline{y} = \underline{y}$. Thus, $\overline{v} = \underline{v}$. In addition, we can get when $\overline{\omega} = \underline{\omega}$ is holds, then $\overline{x} = \underline{x}$, and the reverse is also true. Hence, we have $\overline{x} = \underline{x}$ and $\overline{u} = \underline{u}$. It is shown that both $(\overline{x}, \overline{y}, \overline{u}, \overline{v})$ and $(\underline{x}, \underline{y}, \underline{u}, \underline{v})$ are solutions of the system (1). When condition $\alpha < b_1\gamma$ holds, the system (1) has a unique positive solution $E^*(x^*, y^*, \mu^*, v^*)$. Therefore, $\overline{x} = \underline{x} = x^*$, $\overline{y} = \underline{y} = y^*$, $\overline{u} = \underline{u} = \mu^*$, $\overline{v} = \underline{v} = v^*$.

i.e., $U_1 = V_1 = \lim\limits_{t\to+\infty} x(t) = x^*$, $U_2 = V_2 = \lim\limits_{t\to+\infty} y(t) = y^*$, $U_3 = V_3 = \lim\limits_{t\to+\infty} \mu(t) = \mu^*$, $U_4 = V_4 = \lim\limits_{t\to+\infty} v(t) = v^*$.

The proof of theorem 1 is completed.

4 Global asymptotically stability

Theorem 2 Assume that $\alpha<\min\{b_1\gamma,\ a_{11}\gamma^2\}$, then $E^*(x^*,\ y^*,\ \mu^*,\ v^*)$ is global asymptotically stability.

Proof Define the Lyapunov function

$$V(t)=\delta_1\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+\delta_2\left(y-y^*-y^*\ln\frac{y}{y^*}\right)+\delta_3(\mu-\mu^*)^2+\delta_4(v-v^*)^2,$$

where $\delta_i\,(i=1,\ 2,\ 3,\ 4)$ is the undetermined positive coefficient. We calculate the derivative of $V(t)$ along the positive solution of system (1),

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}&=\delta_1(x-x^*)\left(b_1-a_{11}x-\frac{\alpha}{x+\gamma}+a_{12}y-c_1\mu\right)+\delta_2(y-y^*)(b_2-a_{22}y-c_2v)\\&\quad+2\delta_3(\mu-\mu^*)(-p_1\mu+q_1x)+2\delta_4(v-v^*)(-p_2v+q_2y)\\&=-\left(a_{11}-\frac{\alpha_1}{(x+\gamma)(x^*+\gamma)}\right)\delta_1(x-x^*)^2+\delta_1a_{12}(x-x^*)(y-y^*)-\delta_1c_1(x-x^*)(\mu-\mu^*)\\&\quad-\delta_2a_{22}(y-y^*)^2-\delta_2c_2(y-y^*)(v-v^*)-2\delta_3p_1(\mu-\mu^*)^2+2\delta_3(\mu-\mu^*)(x-x^*)\\&\quad-2\delta_4p_2(v-v^*)^2+2\delta_4q_2(v-v^*)(y-y^*)\\&\leqslant-\left(a_{11}-\frac{\alpha}{\gamma^2}\right)\delta_1(x-x^*)^2+\delta_1a_{12}(x-x^*)(y-y^*)-\delta_1c_1(x-x^*)(\mu-\mu^*)-\delta_2a_{22}(y-y^*)^2\\&\quad-\delta_2c_2(y-y^*)(v-v^*)-2\delta_3p_1(\mu-\mu^*)^2+2\delta_3(\mu-\mu^*)(x-x^*)-2\delta_4p_2(v-v^*)^2+2\delta_4q_2(v-v^*)(y-y^*).\end{aligned}$$

Assume $\delta_1=1$, $\delta_2=\dfrac{a_{12}^2}{4\left(a_{11}-\dfrac{\alpha}{\gamma^2}\right)a_{22}}$, $\delta_3=\dfrac{c_1}{2q_1}$, $\delta_4=\dfrac{a_{12}^2c_2}{8\left(a_{11}-\dfrac{\alpha}{\gamma^2}\right)a_{22}q_2}$, then

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}&\leqslant-(a_{11}-\frac{\alpha}{\gamma^2})\delta_1(x-x^*)^2+\delta_1a_{12}(x-x^*)(y-y^*)-\delta_2a_{22}(y-y^*)^2-2\delta_3p_1(\mu-\mu^*)^2-2\delta_4p_2(v-v^*)^2\\&=-\left[\sqrt{(a_{11}-\frac{\alpha}{\gamma^2})\delta_1}\,(x-x^*)-\sqrt{\delta_2a_{22}}\,(y-y^*)\right]^2-2\delta_3p_1(\mu-\mu^*)^2-2\delta_4p_2(v-v^*)^2.\end{aligned}$$

We can conclude that $\dfrac{\mathrm{d}V}{\mathrm{d}t}\leqslant0$, since $a_{11}-\dfrac{\alpha}{\gamma^2}>0$ (or $\alpha<a_{11}\gamma^2$). Also, $\dfrac{\mathrm{d}V}{\mathrm{d}t}=0$, if and only if $x=x^*$, $y=y^*$, $\mu=\mu^*$, $v=v^*$. It can be seen that when condition $\alpha<\min\{b_1\gamma,\ a_{11}\gamma^2\}$ holds, $E^*(x^*,\ y^*,\ \mu^*,\ v^*)$ is globally asymptotically stable.

The proof of theorem 2 is completed.

5 Numerical simulation

Example Corresponding to system (1), we assume that $b_1=b_2=q_1=p_2=a_{11}=1$, $a_{12}=a_{22}=p_1=q_2=2$, $\alpha=0.2$, $\gamma=c_1=c_2=0.5$. From theorem 2, we know that the positive equilibrium point $E^*(1.227,\ 0.333,\ 0.614,\ 0.667)$ is globally asymptotically stable. The numerical simulation diagram of the solution is as figure 1.

6 Conclusion

In this paper, the influence of additive Allee effect on the dynamical behavior of the system is studied based on the feedback control commensal symbiosis system. We discuss the existence of a unique positive equilibrium for the system (1). The conditions of global attraction and globally asymptotically stable of system (1) are given respectively. The research shows that Allee effect does not affect the stability of the equilibrium

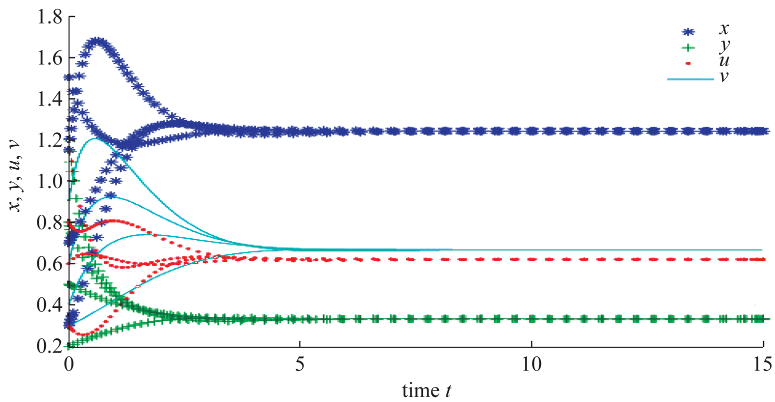

Figure 1 Numerical simulation diagram of the solution of system (1) with initial value $(x, y, u, v)=(0.3, 0.8, 0.3, 0.6)$, $(0.7, 0.5, 1.2, 0.4)$, $(1.15, 1.2, 0.8, 0.9)$ and $(1.5, 0.2, 0.6, 0.3)$

point of the feedback control biased commensal symbiosis system. However, the species with Allee effect can reach equilibrium only when the population is large. Meanwhile, the larger Allee coefficients α and γ, the slower the population growth is, and this also indicates that Allee effect is unfavorable to population growth.

Definitely, system (1) still has some boundary equilibrium points, which are not discussed in this paper. In addition, condition $\alpha<b_1\gamma$ is not a necessary condition for the existence of a unique positive equilibrium point in system (1). We can get different branches of the system (1) by classifying the symbols of the univariate quadratic equation Δ about x. Then, we classify and discuss the dynamic behavior of different branches of the system, which will become more complex, and this can be discussed in subsequent studies.

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