Vol.52, No.3 Journal of Shanghai Normal University (Natural Sciences) Jun., 2023
DOI:10.3969/J.ISSN.1000-5137.2023.03.003

Some irreducible representations of $GL_2(\mathbb{C})$

CHEN Xiaoyu, LAI Yuanxu, LI Zhize

(Mathematics and Science College, Shanghai Normal University, Shanghai 200234, China)

Abstract: Let $\mathbf{G} = GL_2(\mathbb{C})$, and let $\mathbf{B}$ be the standard Borel subgroup of $\mathbf{G}$, and let $\mathbb{C}\mathbf{G}$ (resp. $\mathbb{C}\mathbf{B}$) be the group algebra of $\mathbf{G}$ (resp. $\mathbf{B}$) over the field of complex numbers. For any character θ of $\mathbf{B}$, define the naive induced module $\mathbb{M}(\theta) = \mathbb{C}\mathbf{G} \otimes_{\mathbb{C}\mathbf{B}} \theta$. In this paper, we prove that if θ is antidominant, then $\mathbb{M}(\theta)$ is irreducible. Thus, we give a class of new infinite-dimensional irreducible representations of $GL_2(\mathbb{C})$.

Key words: reductive group; naive induced module; Bruhat decomposition

CLC number: O 186.5 **Document code:** A **Article ID:** 1000-5137(2023)03-0295-08

$GL_2(\mathbb{C})$ 的一些不可约表示

陈晓煜，赖元旭，李支泽

(上海师范大学 数理学院，上海 200234)

摘　要：设 $\mathbf{G} = GL_2(\mathbb{C})$, 并且 $\mathbf{B}$ 是 $\mathbf{G}$ 的标准 Borel 子群，并且 $\mathbb{C}\mathbf{G}$, $\mathbb{C}\mathbf{B}$ 分别是群 $\mathbf{G}$ 和群 $\mathbf{B}$ 的在复数域 $\mathbb{C}$ 上的群代数. 对于任意 $\mathbf{B}$ 的特征标 θ, 定义 $\mathbf{G}$ 的离散诱导模 $\mathbb{M}(\theta) = \mathbb{C}\mathbf{G} \otimes_{\mathbb{C}\mathbf{B}} \theta$. 证明了当 θ 是反支配权时，$\mathbb{M}(\theta)$ 是个不可约表示. 由此给出了一类 $GL_2(\mathbb{C})$ 全新的、无限维的不可约表示.

关键词：简约群；朴素诱导模；Bruhat 分解

Received date: 2023-04-20

Foundation item: Shanghai Sailing Program (17YF1413800); The National Natural Science Foundation of China (11701373)

Biography: CHEN Xiaoyu (1985—), male, associate professor, research area: algebraic groups and quantum groups. E-mail: chenxiaoyu@shnu.edu.cn

引用格式：陈晓煜，赖元旭，李支泽. $GL_2(\mathbb{C})$ 的一些不可约表示 [J]. 上海师范大学学报 (自然科学版), 2023, 52(3): 295–302.

Citation format: CHEN X Y, LAI Y X, LI Z Z. Some irreducible representations of $GL_2(\mathbb{C})$ [J]. Journal of Shanghai Normal University (Natural Sciences), 2023, 52(3):295–302.

1 Introduction

The classification of irreducible representations (up to isomorphism) of a given group is a fundamental problem in the representation theory. Let $\mathbf{G}$ be a connected reductive group over a field $\Bbbk$, e.g. $GL_n(\Bbbk)$, $SL_n(\Bbbk)$. The representation of $\mathbf{G}$ plays a prominent role in various areas of mathematics such as algebraic geometry and number theory. In the case $\Bbbk = \bar{\mathbb{F}}_q$, the algebraic closure of $\mathbb{F}_q$ ($\mathbb{F}_q$ is the finite field of q elements) or $\Bbbk = \mathbb{C}$, the field of complex numbers, the irreducible rational representations of $\mathbf{G}$ was classified by CHEVALLEY in 1958[1]. In 1973, BOREL and TITS classified all finite-dimensional irreducible representations of $\mathbf{G}$ when $\Bbbk$ is an infinite field and thus verified a conjecture of Steinberg [2]. In the case $\Bbbk = \mathbb{F}_q$, the classification of irreducible ordinary representations of $\mathbf{G}$ was given in [3] by DELIGNE and LUSZTIG, and the classification of irreducible modular representations was given in [4] by ROUQUIER and BONNAFE.

Despite the fruitful results mentioned above, little was known about the infinite-dimensional irreducible representations of $\mathbf{G}$ when $\Bbbk$ is an infinite field. In 2014, XI began to study the infinite-dimensional representations of $\mathbf{G}$ when $\Bbbk = \bar{\mathbb{F}}_q$. He constructed such representations (in particular, the infinite-dimensional Steinberg modules) via the union of irreducible representations of the finite groups $\mathbf{G}(\mathbb{F}_{q^r})$[5]. XI showed that the infinite-dimensional Steinberg module is irreducible if the base field is of characteristic zero or characteristic of $\mathbb{F}_q$. In 2015, YANG proved in [6] that the infinite-dimensional Steinberg module is irreducible for any base field when $\Bbbk = \bar{\mathbb{F}}_q$. Let $\mathbf{B}$ be the standard Borel subgroup defined over $\mathbb{F}_q$ and $\Bbbk'$ is another field, and let θ be a character of $\mathbf{B}$, and define $\mathbb{M}(\theta)_{\Bbbk'} = \Bbbk'\mathbf{G} \otimes_{\Bbbk'\mathbf{B}} \theta$. Assume that $\Bbbk = \bar{\mathbb{F}}_q$. In 2019, CHEN and DONG proved in [7] that $\mathbb{M}(\theta)_{\Bbbk'}$ has finite length and determined all composition factors when the characteristic of $\Bbbk'$ is not equal to that of $\bar{\mathbb{F}}_q$, and $\mathbb{M}(\theta)_{\Bbbk'}$ may have infinite length when $\Bbbk' = \bar{\mathbb{F}}_q$. In [8] and [9], CHEN and DONG determined the decomposition of $\mathbb{M}(\mathrm{tr})_{\Bbbk'}$ for any field $\Bbbk'$ when $\mathbb{M}(\theta)_{\Bbbk'}$ has finite length. Recently, CHEN classified all irreducible $\Bbbk\mathbf{G}$-modules with $\mathbf{B}$-stable line when $\Bbbk = \Bbbk' = \bar{\mathbb{F}}_q$[10].

In this paper, we deal with $\Bbbk = \Bbbk' = \mathbb{C}$ and $\mathbf{G} = GL_2(\mathbb{C})$. Let θ be a character of $\mathbf{B}$, we give a sufficient condition for the irreducibility of $\mathbb{M}(\theta) = \mathbb{C}\mathbf{G} \otimes_{\mathbb{C}\mathbf{B}} \theta$. This paper is organized as follows: in section 2, we recall some basic facts of group representations, and give the main result. In section 3, we recall the Bruhat decomposition for $GL_2(\mathbb{C})$ which will be frequently used later. In section 4, we give the proof of the main result.

2 Basic definition and main result

In this section, we recall some basics for group representations and the structure of reductive groups. Let V be a vector space over the field $\mathbb{C}$ of complex numbers, and let $GL(V)$ be the group of invertible linear maps of V, and let G be a group, with identity element 1.

Definition 1 (linear representation). A linear representation of G is a group homomorphism

$$\rho: \ G \to GL(V), \quad g \mapsto \rho(g).$$

For all $v \in V$, $\rho(g)v$ is abbreviated to gv. When ρ is given, we say that V is a representation space of G (or even simply, by abuse of language, a representation of G).

Definition 2 (sub-representation). Let W be a subspace of V, then W is called a sub-representation of V if $gw \in W$ for all $g \in G$ and $w \in W$.

Definition 3 (irreducible representation). Let V be a representation of G, V is said to be irreducible if the only sub-representations of V are $\{0\}$ and V.

We denote by $\mathbb{C}G$ the group algebra of G over $\mathbb{C}$. It is well known that a representation of G is identified with a $\mathbb{C}G$-module, and we will indiscriminately use the terminology "linear representation" or "module".

Definition 4 (induced representation). Let H be a subgroup of G, and let W be a left $\mathbb{C}H$-module. Then the tensor product of the right $\mathbb{C}H$-module $\mathbb{C}G$ and the left $\mathbb{C}H$-module W

$$\mathbb{C}G \underset{\mathbb{C}H}{\otimes} W$$

is a left $\mathbb{C}G$-module which is denoted by $\mathrm{Ind}_H^G W$, and is called the induced representation from $\mathbb{C}H$-module W.

Following the notation above, it is well–known that a basis of the induced representation is given by the following proposition (cf. [11]).

Proposition 1 Suppose that $\{w_i \mid i \in I\}$ is a basis of W and $\{g_j H \mid j \in J\}$ is the set of left cosets of G with respect to H. Then the set

$$\{g_j \otimes w_i \mid i \in I, j \in J\}$$

forms a basis of $\mathrm{Ind}_H^G W$.

One of the most commonly used methods to get a new representation of G is to study the induced representations from some subgroups of G.

Let $\mathbf{G} = GL_2(\mathbb{C})$ and let $\mathbf{B}$ be the standard Borel subgroup of $\mathbf{G}$ (the set of upper-triangular matrixes in $\mathbf{G}$).

Then $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$, where

$$\mathbf{U} = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{C} \right\}, \ \mathbf{T} = \left\{ \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \mid c, d \in \mathbb{C}^{\times} \right\}.$$

Since $\mathbf{U}$ is a normal subgroup of $\mathbf{B}$, we have the natural homomorphism $\pi : \mathbf{B} \to \mathbf{T}$ and $\ker \pi = \mathbf{U}$. Let $\bar{\theta}$ be a homomorphism from group $\mathbf{T}$ into group $\mathbb{C}^{\times}$. Then the pullback of $\bar{\theta}$ by π, denoted by θ for convenience ($\theta = \bar{\theta} \circ \pi$), is a group homomorphism from $\mathbf{B}$ into $\mathbb{C}^{\times}$ i.e. a character of $\mathbf{B}$.

Let $\mathbb{C}_\theta$ be the 1-dimensional space affording the character θ. We are interested in the induced module $\mathbb{C}\mathbf{G} \otimes_{\mathbb{C}\mathbf{B}} \mathbb{C}_\theta$ which is denoted by $\mathbb{M}(\theta)$. A nature question is whether $\mathbb{M}(\theta)$ is an irreducible module.

Definition 5 (antidominant). A character θ of $\mathbf{B}$ is called antidominant if there is an $n \in \mathbb{Z}_{>0}$, such that

$$\theta\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) = t^{-n},$$

for all $t \in \mathbb{C}^{\times}$.

The main result of this paper is the following theorem.

Theorem 1 If θ is antidominant, then $\mathbb{M}(\theta)$ is irreducible.

3 Bruhat decomposition of $GL_2(\mathbb{C})$

For any reductive group of $\mathbf{G}$, one has the Bruhat decomposition (cf. [12, Chapter 8]). In this section we recall the Bruhat decomposition of $GL_2(\mathbb{C})$. In order to get a basis of $\mathbb{M}(\theta)$, We need to understand $\mathbf{G}/\mathbf{B}$. Let $\mathbf{B}\backslash\mathbf{G}/\mathbf{B}$ be the set of double cosets of $\mathbf{G}$ with respect to $\mathbf{B}$.

It is well known that

$$\mathbf{G} = \mathbf{B} \cup \mathbf{B}s\mathbf{B},$$

where $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is called Bruhat decomposition of $GL_2(\mathbb{C})$.

Proposition 2 There is a bijection $\phi : \mathbf{U} \times \{s\} \times \mathbf{B} \to \mathbf{B}s\mathbf{B}$.

Proof Let $b_1 s b_2$ be an arbitrary element of $\mathbf{B}s\mathbf{B}$. Since $\mathbf{B} = \mathbf{U} \rtimes \mathbf{T}$, we have $b_1 = ut$ where $u \in \mathbf{U}, t \in \mathbf{T}$, then $b_1 s b_2 = utsb_2 = us(stsb_2)$, where $us(stsb_2) \in \mathbf{U}s\mathbf{B}$. We define $\phi((u, s, b)) = usb$ which is a bijection.

Combining $\mathbf{G} = \mathbf{B} \cup \mathbf{B}s\mathbf{B}$ and proposition 2. We see that the set of all represent elements of cosets in $\mathbf{G}/\mathbf{B}$ can be written as $\{I_2\} \cup \{us \mid u \in \mathbf{U}\}$. For any $a \in \mathbb{C}$, we set $\epsilon(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \mathbf{U}$.

Proposition 3 For any $a \in \mathbb{C}^\times$ we have

$$s\epsilon(a)s = \epsilon(a^{-1})s \begin{pmatrix} a & 0 \\ 0 & -a^{-1} \end{pmatrix} \epsilon(a^{-1}). \tag{1}$$

Proof First, the left side of (1) equals to

$$s\epsilon(a)s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \in \mathbf{G},$$

and the right side of (1) equals to

$$\begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & -a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix},$$

and thus the result is proved.

4 Proof of the main result

Throughout this section, we assume that θ is antidominant. In order to show that $\mathbb{M}(\theta)$ is an irreducible $\mathbb{C}\mathbf{G}$-module, it is enough to show $\mathbb{M}(\theta) = \mathbb{C}\mathbf{G}.x$ for any $0 \neq x \in \mathbb{M}(\theta)$. Since the set of all representatives of cosets in $\mathbf{G}/\mathbf{B}$ can be written as $\{I_2\} \cup \{us \mid u \in \mathbf{U}\}$ and proposition 1, we see that $\{us \otimes 1 \mid u \in \mathbf{U}\} \cup \{1 \otimes 1\}$ forms a basis of $\mathbb{M}(\theta)$. We abbreviate $us1$ for $us \otimes 1$, 1 for $1 \otimes 1$, then we have

$$x = l1 + \sum_{i=1}^{r} c_i \epsilon(a_i)s1 \text{ where } l, c_i \in \mathbb{C},\ r \in \mathbb{Z}_{>0}.$$

Since $x \neq 0$, l and c_i are not all equal to 0. If all of the c_i are equal to 0, then $x = l1$ with $l \neq 0$ and we get $\mathbb{C}\mathbf{G}.x = \mathbb{C}\mathbf{G}.1 = \mathbb{M}(\theta)$, the proof is finished.

Now we suppose that c_i are not all equal to 0. Since $\theta = \bar{\theta} \circ \pi$ and $\ker \pi = \mathbf{U}$, we have $\epsilon(a).1 = 1$. Since $\{a_i\}$ is a finite set, we can take $a \in \mathbb{C}$ such that

$$0 \neq (1 - \epsilon(a))\,.x = \sum_{i=1}^{r} c_i \left[\epsilon(a_i) - \epsilon(a_i + a)\right] s1 \in \mathbb{C}\mathbf{G}.x\ .$$

Without loss of generality, we can assume that $x = \sum_{i=1}^{r} k_i \epsilon(a_i)s1$ (otherwise we replace x by $(1 - \epsilon(a))\,.x$ for a suitable choice of a).

The following idea is motivated by [13, proposition 5.4].

Lemma 1 If $x = \sum_{i=1}^{r} k_i \epsilon(a_i)s1$, then we have $\sum_{i=1}^{r} k_i s1 \in \mathbb{C}\mathbf{G}.x$.

Proof Let $m_i = k_i s1$, then we have

$$x = \epsilon(a_1)m_1 + \epsilon(a_2)m_2 + \cdots + \epsilon(a_r)m_r \in \mathbb{C}\mathbf{G}.x\ .$$

Using the formula

$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & ad \\ 0 & 1 \end{pmatrix}$$

and noting that

$$\begin{aligned}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\epsilon(a_i)s1 &= \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}1 \\ &= \begin{pmatrix} 1 & ka_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}1 \\ &= \begin{pmatrix} 1 & ka_i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}1 \\ &= \theta\left(\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\right)\epsilon(a_i)^k s1,\end{aligned}$$

we have

$$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}.x = \theta\left(\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\right)\left(\epsilon(a_1)^k m_1 + \epsilon(a_2)^k m_2 + \cdots + \epsilon(a_r)^k m_r\right) \in \mathbb{C}\mathbf{G}.x.$$

Thus, we have

$$\epsilon(a_1)^k m_1 + \epsilon(a_2)^k m_2 + \cdots + \epsilon(a_r)^k m_r \in \mathbb{C}\mathbf{G}.x \text{ for } k = 1,2,\cdots,r. \tag{2}$$

Let $z_i = \epsilon(a_i) \in \mathbb{C}\mathbf{U},\ i = 1,2,\cdots,r$ and $\sigma_k = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant r} z_{i_1}z_{i_2}\cdots z_{i_k} \in \mathbb{C}\mathbf{U},\ k = 1,2,\cdots,r$. Using the identity

$$(z_k - z_1)(z_k - z_2)\cdots(z_k - z_r) = 0,\ k = 1,2,\cdots,r\ ,$$

we have

$$z_k^r - \sigma_1 z_k^{r-1} + \sigma_2 z_k^{r-2} + \cdots + (-1)^r\sigma_r = 0,\ k = 1,2,\cdots,r.$$

Then we obtain the aligns

$$\begin{aligned}&\left(z_1^r - \sigma_1 z_1^{r-1} + \sigma_2 z_1^{r-2} + \cdots + (-1)^r\sigma_r\right)m_1 = 0, \\ &\left(z_2^r - \sigma_1 z_2^{r-1} + \sigma_2 z_2^{r-2} + \cdots + (-1)^r\sigma_r\right)m_2 = 0, \\ &\quad\cdots \\ &\left(z_r^r - \sigma_1 z_r^{r-1} + \sigma_2 z_r^{r-2} + \cdots + (-1)^r\sigma_r\right)m_r = 0.\end{aligned}$$

Adding the equations above, we obtain

$$\begin{aligned}&(-1)^r\sigma_r\left(\sum_{i=1}^r m_i\right) \\ =&\ (-1)^r\sigma_{r-1}\sum_{i=1}^r z_i m_i + (-1)^{r-1}\sigma_{r-2}\sum_{i=1}^r z_i^2 m_i + \cdots + \sigma_1\sum_{i=1}^r z_i^{r-1}m_i - \sum_{i=1}^r z_i^r m_i\ .\end{aligned} \tag{3}$$

Since

$$\sigma_r = \epsilon(a_1)\epsilon(a_2)\cdots\epsilon(a_r) = \epsilon(a_1 + a_2 + \cdots + a_r)$$

is invertible, combining (2) and (3), we obtain

$$m_1 + m_2 + \cdots + m_r \in \mathbb{C}\mathbf{G}.x,$$

and the result is proved.

By lemma 1 we have $\sum_{i=1}^{r} k_i s1 \in \mathbb{C}\mathbf{G}.x$. If $\sum_{i=1}^{r} k_i \neq 0$, then $s1 \in \mathbb{C}\mathbf{G}.x$, thus $\mathbb{C}\mathbf{G}.x = \mathbb{M}(\theta)$.

Now we assume that $\sum_{i=1}^{r} k_i = 0$, it is clear that there is a $c \in \mathbb{C}$ such that $a_i + c \neq 0$ for all a_i. For such c, we have $s\epsilon(c)1.x = \sum_{i=1}^{r} k_i s\epsilon(a_i + c)s1$.

By proposition 3,

$$s\epsilon(a)s = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \epsilon(a^{-1})s\begin{pmatrix} a & 0 \\ 0 & -a^{-1} \end{pmatrix}\epsilon(a^{-1}),$$

and since θ is antidominant, we have $\theta\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = t^{-n}$.

Combining these, we have

$$s\epsilon(c)1.x = \sum_{i=1}^{r} k_i s\epsilon(a_i + c)s1 = \theta\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\sum_{i=1}^{r} k_i(a_i + c)^{-n}\epsilon((a_i + c)^{-1})s1.$$

Now, if there exists $c \in \mathbb{C}$ such that $\sum_{i=1}^{r} k_i(a_i + c)^{-n} \neq 0$, then we can replace k_i with $k_i(a_i + c)^{-n}$, and the proof is finished. To see this, it is enough to show the following lemma.

Lemma 2 Let $\{f_i(x) = (a_i + x)^{-n} \mid i = 1, \cdots, r\}$ be a finite set of complex value functions, then $\{f_i(x)\}$ are linearly independent.

Proof The Wronsky determinant of $\{f_i(x)\}$ is

$$\begin{aligned} W(x) &= \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_r(x) \\ f_1'(x) & f_2'(x) & \cdots & f_r'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{r-1}(x) & f_2^{r-1}(x) & \cdots & f_r^{r-1}(x) \end{vmatrix} \\ &= C\begin{vmatrix} (a_1 + x)^{-n} & (a_2 + x)^{-n} & \cdots & (a_r + x)^{-n} \\ (a_1 + x)^{-(n+1)} & (a_2 + x)^{-(n+1)} & \cdots & (a_r + x)^{-(n+1)} \\ \vdots & \vdots & & \vdots \\ (a_1 + x)^{-(n+r-1)} & (a_2 + x)^{-(n+r-1)} & \cdots & (a_r + x)^{-(n+r-1)} \end{vmatrix}, \end{aligned}$$

where

$$C = \prod_{i=1}^{r-1}\prod_{k=0}^{i-1}(-1)^i(n + k).$$

Let $x = 0$, we have

$$W(0) = C \begin{vmatrix} a_1^{-n} & a_2^{-n} & \cdots & a_r^{-n} \\ a_1^{-(n+1)} & a_2^{-(n+1)} & \cdots & a_r^{-(n+1)} \\ \vdots & \vdots & & \vdots \\ a_1^{-(n+r-1)} & a_2^{-(n+r-1)} & \cdots & a_r^{-(n+r-1)} \end{vmatrix} = D \begin{vmatrix} 1 & 1 & \cdots & 1 \\ b_1 & b_2 & \cdots & b_r \\ \vdots & \vdots & & \vdots \\ b_1^{r-1} & b_2^{r-1} & \cdots & b_r^{r-1} \end{vmatrix},$$

where

$$b_i = a_i^{-1},\ D = \prod_{j=1}^{r} b_j^n \prod_{i=1}^{r-1} \prod_{k=0}^{i-1} (-1)^i (n+k).$$

Since $a_i \neq a_j\ (i \neq j)$, $b_i \neq b_j$, we have $W(0) \neq 0$, thus $f_1(x), f_2(x), \cdots, f_r(x)$ are linearly independent.

Thus, by lemma 2 there exists $c \in \mathbb{C}$, such that $\sum_{i=1}^{r} k_i(a_i + c)^{-n} \neq 0$. then the theorem is proved.

References:

[1] CHEVALLEY S C. Classification des Groupes de Lie Algébriques [M]. Paris: Secreétariat Mathématique, 1958.

[2] BOREL A, TITS J. Homomorphismes “abstraits” de groupes algebriques simples [J]. Annals of Mathematics, 1973, 97(3): 499–571.

[3] DELIGNE P, LUSZTIG G. Representations of reductive groups over finite fields [J]. Annals of Mathematics, 1976, 103(1): 103–161.

[4] BONNAFÉ C, ROUQUIER R. Catégories dérivées et variétés de Deligne-Lusztig [J]. Publications Mathématiques de l’Institut des Hautes études Scientifiques, 2003, 97(1): 1–59.

[5] XI N H. Some infinite dimensional representations of reductive groups with Frobenius maps [J]. Science China Mathematics, 2014, 57(6): 1109–1120.

[6] YANG R T. Irreducibility of infinite dimensional Steinberg modules of reductive groups with Frobenius maps [J]. Journal of Algebra, 2019, 533: 17–24.

[7] CHEN X Y, DONG J B. Abstract-induced modules for reductive algebraic groups with Frobenius maps [J]. International Mathematics Research Notices, 2022, 2022(5): 3308–3348.

[8] CHEN X Y, DONG J B. The permutation module on flag varieties in cross characteristic [J]. Mathematische Zeitschrift, 2019, 293(1): 475–484.

[9] CHEN X Y, DONG J B. The decomposition of permutation module for infinite Chevalley groups [J]. Science China Mathematics, 2021, 64(5): 921–930.

[10] CHEN X Y. Irreducible modules of reductive groups with B-stable line [J]. arXiv: 2011.04115, 2020.

[11] SERRE J P. Linear Representations of Finite Groups [M]. New York: Springer, 1977.

[12] CARTER R W. Simple Groups of Lie Type [M]. London: John Wiley & Son Ltd, 1972.

[13] PUTMAN A, SNOWDEN A. The Steinberg representation is irreducible [J]. Duke Mathematical Journal, 2023, 172(4): 775–808.

(责任编辑：冯珍珍)