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解平行四边形板弯曲问题的二元 *B* 样条有限元法*

Finite Element Method with Bivariate *B* Splines in Solving a Bending Problem of Parallelogram Boards

刘焕文

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摘要　将文献 [1]以二元二次 B 样条函数为基底,求解矩形薄板弯曲问题的二元 B 样条有限元的方法推广到用于求解平行四边形板弯曲问题.结果表明: 该方法系数矩阵每行的非零元仅 21个,相对于朱明权和 Chui C. K. 等的张量积型样条有限元方法,计算量与存贮量都大大节约.

关键词　平行四边形板　弯曲问题　二元 B 样条有限元

Abstract　A finite element method with bivariate B spline is given in [1] to solve a bending problem of rectangular boards based on the binary quadric B spline functions. we generalize the method to the case of parallelogram boards. It turns out that the method greatly reduces the computations and memory as comparied with the finite element method with splines of tensor product type of Zhu Mingquan and Chui C. K. etc.

Key words　parallelogram boards, bending problem, finite element with bivariate B splines
中图法分类号　O 241

1　坐标变换下的弯曲问题

对各向同性、均质等厚度的平行四边形板 R(如图 1所示,原点及 $P_i(x_i, y_i), i = 1, 2, 3$为其四个角点),其带固支边界条件的弯曲问题,即下述双调和方程的齐次第一边值问题:

$$\begin{cases} \triangle^2 u = f/D. & \text{在 } R \text{ 内} \\ u = \dfrac{\partial R}{\partial n} = 0 & \partial R \text{ 上} \end{cases} \tag{1.1}$$

其中 $D = \dfrac{Eh^3}{12(1-\_^2)}$ 为板的抗弯刚度,h为板厚,_为 Poisson比,$0 < \_ < 1$,f为板上的垂直荷载(本文仅考虑均布荷载,即 $f = q =$ const),n为边界 ∂R的外法线方向.

问题(1.1)相应的 Galerkin变分形式为:

$$A(u,v) = \langle f,v\rangle, \qquad \forall u \in \overset{0}{H}_2(R) \tag{1.2}$$

其中

$$A(u,v) = D\iint_z \{\triangle u \triangle v + (1-\_)[2\frac{\partial u}{\partial x\partial y}\frac{\partial v}{\partial x\partial y} - \frac{\partial u}{\partial x^2}\frac{\partial v}{\partial y^2} - \frac{\partial u}{\partial y^2}\frac{\partial v}{\partial x^2}]\} dxdy$$

$$\langle f,v\rangle = \iint_z fv\,dxdy.$$

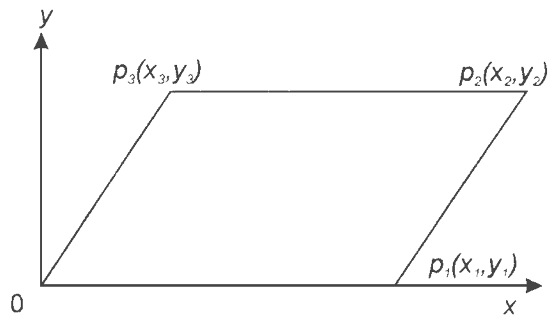

图 1 Fig. 1

熟知[2],$A(u,v)$是 $\overset{0}{H}_2(R)$上对称、连续和强制的双线性泛函,故由 Lax-Milgram定理知变分问题(1.2)存在唯一解 u.

为使问题(1.2)简化,运用文献 [3]的方法,引入坐标变换:

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$$\begin{cases} x = x_1 Y + x_3 Z \\ y = y_3 Z \end{cases} \tag{1.3}$$

或

$$\begin{cases} Y = \frac{1}{x_1} x - \frac{x_3}{x_1 y_3} y \\ Z = \frac{1}{y_3} y \end{cases}$$

易知,此变换将 R 变换为单位正方形 $\Omega = \{(Y, Z): 0 \leqslant Y \leqslant 1, 0 \leqslant Z \leqslant 1\}$,且由坐标 (x, y) 变换为坐标 (Y, Z) 的 Jacobi 矩阵的行列式为

$$J = \det \begin{bmatrix} \frac{\partial x}{\partial Y} & \frac{\partial y}{\partial Y} \\ \frac{\partial x}{\partial Z} & \frac{\partial y}{\partial Z} \end{bmatrix} = \begin{vmatrix} x_1 & 0 \\ x_3 & y_3 \end{vmatrix} = x_1 y_3 > 0$$

记

$$u^*(Y, Z) = u(x_1 Y + x_3 Z, y_3 Z),$$
$$v^*(Y, Z) = v(x_1 Y + x_3 Z, y_3 Z).$$

由复合函数求导的连锁规则易得

$$u_x = \frac{1}{x_1} u^*_Y,$$
$$u_y = -\frac{x_3}{x_1 y_3} u^*_Y + \frac{1}{y_3} u^*_Z$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{x_1^2} \frac{\partial^2 u^*}{\partial Y^2}$$
$$\frac{\partial^2 u}{\partial y^2} = \left(\frac{x_3}{x_1 y_3}\right)^2 \frac{\partial^2 u^*}{\partial Y^2} - 2\frac{x_3}{x_1 y_3^2} \frac{\partial^2 u^*}{\partial Y \partial Z} + \frac{1}{y_3^2} \frac{\partial^2 u^*}{\partial Z^2}$$
$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{x_3}{x_1^2 y_3} \frac{\partial^2 u^*}{\partial Y^2} + \frac{1}{x_1 y_3} \frac{\partial^2 u^*}{\partial Y \partial Z}$$

将 (1.3) 代入 (1.2),则得:

$$A(u, v) = \frac{D}{J^2} \iint_\Omega \{(x_3^2 + y_3^2)^2 \frac{\partial^2 u^*}{\partial Y^2} \frac{\partial^2 v^*}{\partial Y^2} - 2x_1 x_3 (x_3^2 + y_3^2) \frac{\partial^2 u^*}{\partial Y^2} \frac{\partial^2 v^*}{\partial Y \partial Z} - 2x_1 x_3 (x_3^2 + y_3^2) \frac{\partial^2 u^*}{\partial Y \partial Z} \frac{\partial^2 v^*}{\partial Y^2} + x_1^2 x_3^2 \frac{\partial^2 u^*}{\partial Y^2} \frac{\partial^2 v^*}{\partial Z^2} + x_1^2 x_3^2 \frac{\partial^2 v^*}{\partial Y^2} \frac{\partial^2 u^*}{\partial Z^2} + 2x_1^2 (2x_3^2 + y_3^2) \frac{\partial^2 u^*}{\partial Y \partial Z} \frac{\partial^2 v^*}{\partial Y \partial Z} - 2x_1^3 x_3 \left(\frac{\partial^2 u^*}{\partial Y \partial Z} \frac{\partial^2 v^*}{\partial Z^2} + \frac{\partial^2 u^*}{\partial Z^2} \frac{\partial^2 v^*}{\partial Y \partial Z}\right) + x_1^4 \frac{\partial^2 u^*}{\partial Z^2} \frac{\partial^2 v^*}{\partial Z^2} + J^2 \_ \left[\frac{\partial^2 u^*}{\partial Y^2} \frac{\partial^2 v^*}{\partial Z^2} + \frac{\partial^2 u^*}{\partial Z^2} \frac{\partial^2 v^*}{\partial Y^2} - 2\frac{\partial^2 u^*}{\partial Y \partial Z} \frac{\partial^2 v^*}{\partial Y \partial Z}\right]\} \mathrm{d}Y\mathrm{d}Z \triangleq A^*(u^*, v^*)$$

$$\langle f, v \rangle = J \iint_\Omega f(x_1 Y + x_3 Z, y_3 Z) v(x_1 Y + x_3 Z, y_3 Z) \mathrm{d}Y\mathrm{d}Z = J \iint_\Omega f^*(Y, Z) v^*(Y, Z) \mathrm{d}Y\mathrm{d}Z \triangleq \langle f^*, v^* \rangle^*.$$

又文献 [3] 证明了固支边界条件

$$u(x, y)|_{\partial\Omega} = \frac{\partial R(x, y)}{\partial n}|_{\partial\Omega} = 0$$

经过坐标变换 (1.3) 后,恰好变成完全相同的形式:

$$u^*(Y, Z)|_{\partial\Omega} = \frac{\partial u^*(Y, Z)}{\partial n}|_{\partial\Omega} = 0$$

故问题 (1.2) 完全等价于:

$$A^*(u^*, v^*) = \langle f^*, v^* \rangle^*, \ \forall v^* \in \overset{0}{H}_2(\Omega) \tag{1.4}$$

显然,坐标变换下双线性泛函的强制性、连续性和对称性仍然保持,故由 Lax-Milgram 定理知存在唯一解.

2 二元二次 B 样条有限元解

以 $\triangle_{mn}^{(2)}$ 表 Ω 的均匀 (II) 型三角部分,如图 2.它将 Ω 分成 $4mn$ 个小三角形单元.以后总记 $h_x = \frac{1}{m}$, $h_y = \frac{1}{n}$.定义 $\triangle_{mn}^{(2)}$ 上的二元二次 C^1 样条函数空间 $S_2^1(\triangle_{mn}^{(2)})$ 由满足如下条件的所有 $s(Y, Z)$ 组成:

i) $s \in C^1(\Omega)$;

ii) s 在 $\triangle_{mn}^{(2)}$ 的每一三角形上为二次完全多项式.

进一步定义:

$$S_2^{1,1}(\triangle_{mn}^{(2)}) = \{s \in S_2^1(\triangle_{mn}^{(2)}): s|_{\partial\Omega} = \frac{\partial s}{\partial n}|_{\partial\Omega} = 0\}$$

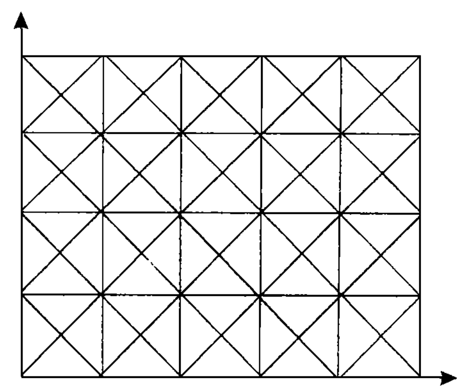

图 2 Fig. 2

引理[3] $\dim S_2^{1,1}(\triangle_{mn}^{(2)}) = (m-2)_+ (n-2)_+$,其空间 $S_2^{1,1}(\triangle_{mn}^{(2)})$ 的一组 B 样条基函数为 $\{B_{ij}(Y, Z)\}$, $(i, j) \in \wedge\}$, $\wedge = \{(i, j): i = 2, \cdots, m-1, j = 2, \cdots, n-1\}$,其中 $B_{ij}(Y, Z) = B(mY - i + \frac{1}{2}, nZ - j + \frac{1}{2})$,而 $B(Y, Z) \in C^1(\Omega)$,其局部支集为以 $(\pm\frac{3}{2}, \pm\frac{3}{2})$, $(\pm\frac{3}{2}, \mp\frac{1}{2})$, $(\pm\frac{1}{2}, \pm\frac{3}{2})$ 及 $(\pm\frac{1}{2}, \mp\frac{3}{2})$ 为顶点的八边形.若记 $B(Y, Z)$ 在图 3 中编号为 i ($i = 1, 2, \cdots, 25$) 的子块 R_i 上的表示为 $p_i(Y, Z)$,则

$$p_1(Y, Z) = \frac{1}{2}(1 - Y^2 - Z^2)$$

$$p_2(Y,Z) = \frac{1}{8}(5 - 4Y - 4Z^2)$$

$$p_6(Y,Z) = \frac{1}{2}(Y - \frac{3}{2})^2 - \frac{1}{4}(Y + Z - 1)^2$$

$$p_7(Y,Z) = \frac{1}{2}(Y - \frac{3}{2})^2$$

$$p_9(Y,Z) = (1 - \frac{1}{2}Y - \frac{1}{2}Z)^2$$

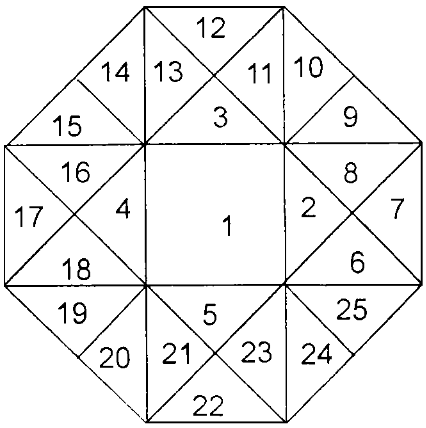

图 3 Fig. 3

其余 $p_i(Y,Z)$ 可由 $B(Y,Z)$ 关于直线 $Y=0, Z=0, Y\pm Z=0$ 的对称性唯一确定.

为着下面计算的需要与方便,我们将 $B(Y,Z)$ 的三个二阶导数在其八边形支集上的值 (为分片常数) 计算出来,开列如图 4.

注意到 $S_2^{1,1}(\triangle_{mn}^{(2)})$ 可嵌入 $\overset{0}{H_2}(\Omega)$, 为使问题 (1. 4) 离散化,本文取 $S_2^{1,1}(\triangle_{mn}^{(2)})$ 为试探函数空间,寻求问题 (1. 4) 在 $S_2^{1,1}(\triangle_{mn}^{(2)})$ 中的解 u^*,即 u^* 满足

$$A^*(u^*, v^*) = \langle f^*, v^* \rangle^*, \ \forall u^* \in S_2^{1,1}(\triangle_{mn}^{(2)}) \tag{2. 1}$$

我们称 u^* 为二元 B 样条有限元解.

不难知道, (2. 1) 等价于

$$A^*(u^*, B_{rt}) = \langle f^*, B_{rt} \rangle^*, \ (r,t) \in \wedge \tag{2. 2}$$

令

$$u^* = \sum_{i=2}^{m-1}\sum_{j=2}^{n-1} C_{ij} B_{ij}(Y,Z) \tag{2. 3}$$

代入 (2. 2),则得如下线性代数方程组:

$$\sum_{i=2}^{m-1}\sum_{j=2}^{n-1} C_{ij} A^*(B_{ij}, B_{rt}) = \langle f^*, B_{rt} \rangle^*, \ (v,t) \in \wedge \tag{2. 4}$$

注意到对 $0 \leqslant i \leqslant 2, 0 \leqslant j \leqslant 2, i+j=2$,有 $\frac{\partial^2}{\partial Y \partial Z} B_{rt}(Y,Z) = \frac{\partial^2}{\partial Y \partial Z} B(mY - r + \frac{1}{2}, nZ - t + \frac{1}{2}) = m^i n^j \frac{\partial^2}{\partial x^i \partial y^j} B(x,y)|_{x=mY-r+\frac{1}{2}, y=nZ-t+\frac{1}{2}}$

于是利用图 4所列的 $\frac{\partial^2 B(Y,Z)}{\partial Y^2}$、$\frac{\partial^2 B(Y,Z)}{\partial Z^2}$、$\frac{\partial^2 B(Y,Z)}{\partial Y \partial Z}$ 的值,经较细致的演算, (2. 4) 可写成如下矩阵形式 (假定 $m,n \geqslant 7$):

$$AC = \frac{4J^3}{Dh_x h_y} F$$

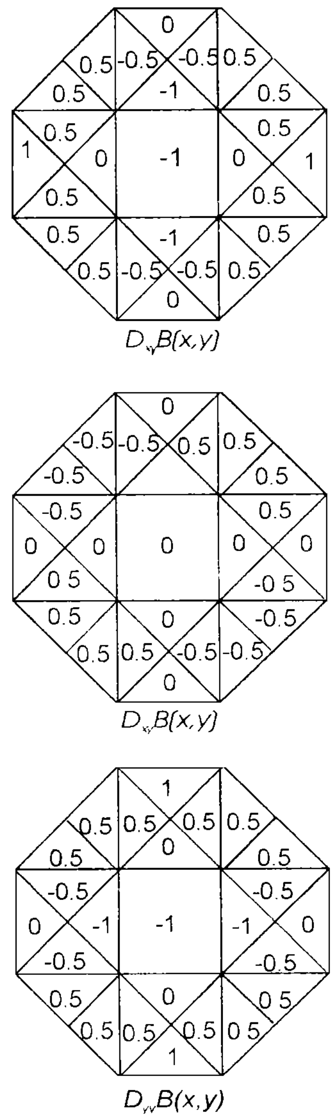

图 4 Fig. 4

其中

$C = (C_{22}, C_{23}, \cdots, C_{2n-1}; C_{32}, C_{33}, \cdots, C_{3n-1}; C_{m-12}, C_{m-13}, \cdots, C_{m-1n-1})^T$,

$F = \frac{J}{6} f h_x h_y (1, 4, 6, 6, \cdots, 6, 6, 4, 1; \cdots; 1, 4, 6, 6, \cdots, 6, 6, 4, 1)^T$,

$A = (x_3^2 + y_3^2)^2 m^4 A_1 - 4x_1^3 x_3 m^3 n A_2 + 2x_1^2 (3x_3^2 +$

$y_3^2)m^2n^2A_3 - 4x_1^2x_3mn^3A_4 + x_1^4n^4A_5$

$$A_i = \begin{bmatrix} B_{i1} & B_{i2} & B_{i3} & & & & \\ B_{i2}^T & B_{i1} & B_{i2} & B_{i3} & & & \\ B_{i3}^T & B_{i2}^T & B_{i1} & B_{i2} & B_{i3} & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & B_{i3}^T & B_{i2}^T & B_{i1} & B_{i2} & B_{i3} & \\ & & B_{i3}^T & B_{i2}^T & B_{i1} & B_{i2} & \\ & & & B_{i3}^T & B_{i2}^T & B_{i1} & \end{bmatrix}$$

其中 B_{ij}^T 表 B_{ij} 的转置，B_{ij} 均为 $n-2$ 阶方阵，具体如下

$$B_{11} = \begin{bmatrix} 12 & 5 & 1 & & & & \\ 5 & 12 & 5 & 1 & & & \\ 1 & 5 & 12 & 5 & 1 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & 1 & 5 & 12 & 5 & 1 \\ & & & 1 & 5 & 12 & 5 \\ & & & & 1 & 5 & 12 \end{bmatrix}.$$

$$B_{12} = -\begin{bmatrix} 7 & 4 & 1/2 & & & & \\ 4 & 7 & 4 & 1/2 & & & \\ 1/2 & 4 & 7 & 4 & 1/2 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & 1/2 & 4 & 7 & 4 & 1/2 \\ & & & 1/2 & 4 & 7 & 4 \\ & & & & 1/2 & 4 & 7 \end{bmatrix}.$$

$$B_{13} = \begin{bmatrix} 1 & 3/2 & & & & \\ 3/2 & 1 & 3/2 & & & \\ & 3/2 & 1 & 3/2 & & \\ & \ddots & \ddots & \ddots & & \\ & & 3/2 & 1 & 3/2 & \\ & & & 3/2 & 1 & 3/2 \\ & & & & 3/2 & 1 \end{bmatrix}.$$

$$B_{22} = \begin{bmatrix} 0 & -2 & & & & \\ 2 & 0 & -2 & & & \\ & 2 & 0 & -2 & & \\ & \ddots & \ddots & \ddots & & \\ & & 2 & 0 & -2 & \\ & & & 2 & 0 & -2 \\ & & & & 2 & 0 \end{bmatrix}.$$

$$B_{31} = \begin{bmatrix} 4 & -1 & -1 & & & & \\ -1 & 4 & -1 & -1 & & & \\ -1 & -1 & 4 & -1 & -1 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & -1 & -1 & 4 & -1 & -1 \\ & & & -1 & -1 & 4 & -1 \\ & & & & -1 & -1 & 4 \end{bmatrix}.$$

而 A_i ($i = 1, 2, \cdots, 5$) 以块计均为 $m-2$ 阶方阵：

$$B_{32} = \begin{bmatrix} -1 & 0 & 1/2 & & & & \\ 0 & -1 & 0 & 1/2 & & & \\ 1/2 & 0 & -1 & 0 & 1/2 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & 1/2 & 0 & -1 & 0 & 1/2 \\ & & & 1/2 & 0 & -1 & 0 \\ & & & & 1/2 & 0 & -1 \end{bmatrix}.$$

$$B_{33} = \begin{bmatrix} -1 & 1/2 & & & & \\ 1/2 & -1 & 1/2 & & & \\ & 1/2 & -1 & 1/2 & & \\ & \ddots & \ddots & \ddots & & \\ & & 1/2 & -1 & 1/2 & \\ & & & 1/2 & -1 & 1/2 \\ & & & & 1/2 & -1 \end{bmatrix}.$$

$$B_{42} = \begin{bmatrix} 0 & -2 & 1 & & & & \\ 2 & 0 & -2 & 1 & & & \\ -1 & 2 & 0 & -2 & 1 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & 0 & -2 & 1 \\ & & & -1 & 2 & 0 & -2 \\ & & & & -1 & 2 & 0 \end{bmatrix}.$$

$$B_{51} = \begin{bmatrix} 12 & -7 & 1 & & & & \\ -7 & 12 & -7 & 1 & & & \\ 1 & -7 & 12 & -7 & 1 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & 1 & -7 & 12 & -7 & 1 \\ & & & 1 & -7 & 12 & -7 \\ & & & & 1 & -7 & 12 \end{bmatrix}.$$

$$B_{52} = \begin{bmatrix} 5 & -4 & 3/2 & & & & \\ -4 & 5 & -4 & 3/2 & & & \\ 3/2 & -4 & 5 & -4 & 3/2 & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & 3/2 & -4 & 5 & -4 & 3/2 \\ & & & 3/2 & -4 & 5 & -4 \\ & & & & 3/2 & -4 & 5 \end{bmatrix}.$$

另外，$B_{23} = -\frac{1}{2}B_{22}$，$B_{42} = -B_{32}$，其余未写出的 B_{ij} 为零矩阵．

容易看出，A 的半带宽仅为 2，而它的子块矩阵亦为带状矩阵，半带宽至多为 2．故 G 的每行最多只有 21 个非零元．而张量积样条有限元法文献 [3, 5] 的系数总矩阵每行有 49 个非零元．相比之下本方法无论其存贮量与计算量都大大节约．

3 算例

例 1 考虑文献 [6] 中曾给出的一个例子，即如图 5 所示均布荷载（密度为 P）下四周简支板的挠

度.假定边长均为 a,其两边夹角的最小者为 60°. 我们分别用网格 8×8, 12×12, 16×16, 20×20及 24×24进行了计算, 表 1中列出了板中心挠度的准确值及我们所算得的数值解. 从表 1的比较可以看出, 二元 B 样条有限元法的精度是相当高的.

表 1 均布荷载下四周简支菱形板在不同网格下中心点处挠度 WPa^4/D及与准确解的比较

Table 1 At center point of the distinct mesh of lozenge boards which is all around simplify branch under well-distribution load, comparison of deflection WPa^4/D and accuracy solution

8×8	12×12	16×16	20×20	24×24	准确解 Accuracy solution
0.0023655	0.0024314	0.0024640	0.0024834	0.0024963	0.0025600

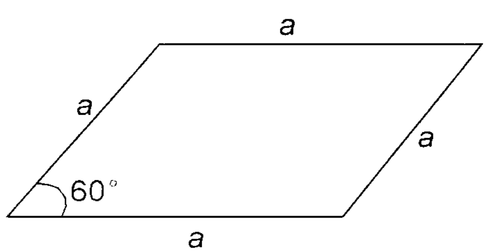

图 5 均布荷载下四周简支菱形板

Fig. 5 Lozenge boards which is all around simplify branch under well-distribution load

例 2 考虑如图 6所示的均布荷载下 (密度为 P) 四周简支的平行四边形板的中心挠度. 为了试验我们的新方法, 我们计算了各种角度, 其相应的对比数值解 (而非标准解) 见经典的 Timonshenko和 Woinowsky-urieger[7]. 表 2给出了本文方法的数值解与对比数值解的比较. 从比较可知, 两组结果在各种角度下都是十分接近的.

表 2 平行四边形板在各种角度下用本文方法计算得到的中心处挠度 WPa^4/D及其对比

Table 2 At center point of the distinct angle of parallelogram boards, deflection WPa^4/D which are calculated with this paper's method compare with others

T	m	本文方法解 Solution of this method	对比解 Numerical solution of comparison
0°	2	0.01012	0.01013
30°	2.02	0.02343	0.01046
45°	2	0.00843	0.00938
60°	2	0.00503	0.00796
75°	2	0.00081	0.00094

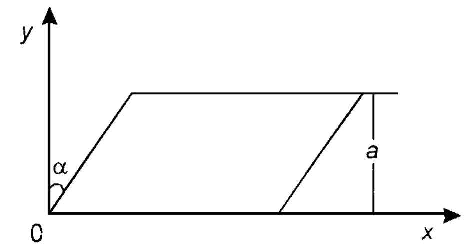

图 6 均布荷载下四周简支的平行四边形板

Fig. 6 Parallelogram boards which is all around simplify branch under well-distribution load

参考文献

1 刘焕文, 何登旭. 解矩形薄板弯曲问题的二元 B 样条有限元法. 应用数学, (待发表).

2 姜礼尚, 庞之坦. 有限元方法及其理论基础. 人民教育出版社, 1982.

3 朱明权. 解任意四边形板弯曲问题的样条有限元法. 计算数学, 1987, 9 (1): 23~42.

4 Chui C K, Schumaker L L, Wang R H. On spaces of piecewise polynomials with boundary condition Ⅲ Type-2 triangulation. In: second edmonton conference on approximation theory. CMS, Conference Proccedings, 1982, 3: 67~80.

5 石钟慈. 样条有限元. 计算数学, 1979, 1 (1): 50~72.

6 Salvadori M G, Baron M L. Numerical methods in engineering. Prentice-Hall, 1967.

7 Timonshenko S, Woinowsky-Krieger S. Theory of plates and shells. 2nd Edn. McGraw-Hill, New York, 1959.

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