网络优先数字出版时间:2014-6-16
网络优先数字出版地址:http://www.cnki.net/kcms/doi/10.13656/j.cnki.gxkx.20140616.003.html
广西科学 Guangxi Sciences 2014,21(3):287～289

The Chromatic Number of the Square of Kneser Graph *KG*(11,5)*

Kneser 图 *KG*(11,5)平方图的色数

XU Xiao-dong[1], LIANG Mei-lian[2], SHAO Ze-hui[3,4]

许晓东[1],梁美莲[2],邵泽辉[3,4]

(1. Guangxi Academy of Sciences, Nanning, Guangxi, 530007, China; 2. School of Mathematics and Information Science, Guangxi University, Nanning, Guangxi, 530004, China; 3. Key Laboratory of Pattern Recognition and Intelligent Information Processing, Institutions of Higher Education of Sichuan Province, Chengdu, Sichuan, 610106, China; 4. School of Information Science and Technology, Chengdu University, Chengdu, Sichuan, 610106, China)

(1.广西科学院,广西南宁　530007;2.广西大学数学与信息科学学院,广西南宁　530004;3.四川省高校模式识别与智能信息处理重点实验室,四川成都　610106;4.成都大学信息科学与技术学院,四川成都　610106)

Abstract: The Kneser graph $KG(n,k)$ is the graph whose vertex set consists of all k-subsets of an n-set, and two vertices are adjacent if and only if they are disjoint. The square G^2 of a graph G is defined on the vertex set of G such that distinct vertices within distancetwo in G are joined by an edge. By theoretical analysis and computer search, we obtain that $8 \leqslant \chi(KG^2(11,5)) \leqslant 10$, which improves the known lower bound 7 and upper bound 12, and that $10 \leqslant \chi(KG^2(13,6)) \leqslant 16$.

Key words: chromatic number, Kneser graph, square graph

摘要:Kneser 图 $KG(n,k)$ 的顶点集包括一个 n 元集的所有 k 元子集,其中的任意两个顶点相邻当且仅当它们对应的子集不相交.一个图 G 的平方图 G^2 的顶点集与 G 的顶点集相同,在 G^2 中两个顶点之间有边当且仅当它们在 G 中的距离不超过 2.通过理论分析和计算机搜索,得到 $8 \leqslant \chi(KG^2(11,5)) \leqslant 10$,$10 \leqslant \chi(KG^2(13,6)) \leqslant 16$,其中前一个结论改进了已知的下界 7 和上界 12.

关键词:色数　Kneser 图　平方图

中图分类号:O175　　**文献标识码**:A　　**文章编号**:1005-9164(2014)03-0287-03

In this note, we shall only consider graphs without multiple edges or loops. For a positive integer n, the set $\{1,2,\cdots,n\}$ is denoted by $[n]$. For a graph $G=(V,E)$, the distance $d_G(u,v)$ between two vertices $u,v \in V$ is the length of a shortest path connecting them. The graph G^2, called the square of G, is defined on V such that two vertices u and v are adjacent in G^2 if mol only if $1 \leqslant d_G(u,v) \leqslant 2$. For $n > 2k > 0$, the Kneser graph $KG(n,k)$ is the graph whose vertex set consists of all k-subsets of the set $[n]$, of which any two vertices A and B are adjacent if mol only if $A \cap B = \varnothing$.

A proper coloring of the vertices of a graph $G=(V,E)$ is a map $f: V \to N$, where adjacent vertices receive distinct colors in N. The chromatic number $\chi(G)$ is the minimum number of colors needed for a

收稿日期:2013-04-07
修回日期:2013-09-10
作者简介:许晓东(1974-),男,研究员,主要从事图论和组合数论研究。
*国家自然科学基金项目(11361008)资助。

proper coloring of G. A graph G is said to be k chromatic if $\chi(G)=k$, and k-colorable if $\chi(G)\leqslant k$.

The chromatic number of Kneser graph was studied by many researchers. In 1955, Kneser[1] proposed a conjecture that $\chi(KG(n,k))=n-2k+2$, which was proved by Lovász[2] and Bárány[3] in 1978. Later a purely combinatorial proof was obtained by Matoušek[4] using Tucker's lemma.

The Kneser graph has some extensions and generalizations, of which the coloring problems also have been considered. For example, Ziegler[5] studied coloring problem of generalized Kneser graphs on hypergraph. Araujo et al[6] researched on coloring problem of some geometric type Kneser graphs.

Recently, the chromatic number of the square of Kneser graph seems interesting. Kim and Nakprasit[7] showed that $\chi(KG^2(2k+1,k))\leqslant 4k$ when k is odd and $\chi(KG^2(2k+1,k))\leqslant 4k+2$ when k is even. In particular, $\chi(KG^2(7,3))=6$ and $11\leqslant\chi(KG^2(9,4))\leqslant 18$. Jun-Yo Chen et al[8] improved the upper bounder for $\chi(KG^2(2k+1,k))$ from $4k$ to $3k+2$. Furthermore, they showed that $\chi(KG^2(9,4))\leqslant 12$. In a reference[9], it was proved that $\chi(KG^2(9,4))=11$. Thus, for $k=1,2,3,4$, the values of $\chi(KG^2(2k+1,k))$ were determined exactly.

The value of $\chi(KG^2(11,5))$ is still unknown. Recently, P. Lakin wrote a presentation discussing the chromatic number of some square Kneser graphs, especially $\chi(KG^2(11,5))$ and provided that $\chi(KG^2(11,5))\leqslant 12$, which is available at http://www.bruce-shapiro.net/math382/Projects/content/Lakin-Presentation.pdf.

In this note, we give a computer-assist proof showing that $8\leqslant\chi(KG^2(11,5))\leqslant 10$.

1 Structures of $KG^2(11,5)$ and a lower bound for $\chi(KG^2(11,5))$

Firstly, we have the following simple lower bound for $\chi(KG^2(2k+1,k))$.

Theorem 1 $\chi(KG^2(2k+1,k))\geqslant k+2$.

Proof In $KG^2(2k+1,k)$, it is not difficult to see that vertex set $\{12\cdots\{k-1\}i \mid k\leqslant i\leqslant 2k+1\}$ is a $k+2$-clique. So $\chi(KG^2(2k+1,k))\geqslant k+2$.

By computer search, the following fact are established.

Fact 1 (1) The independent number of $KG^2(11,5)$ is 66.

(2) There are exactly 5040 independent sets of order 66 in $KG^2(11,5)$.

By Theorem 1 we have that $\chi(KG^2(11,5))\geqslant 7$. By the above Fact 1, it is easy to see that

Lemma 1 $\chi(KG^2(11,5))=7$ if and only if there exist 7 disjoint independent sets of order 66 in $KG^2(11,5)$.

Now we can construct a graph G as follows. $V(G)$ is the set of independent set of order 66. Two vertices in G are joint by an edge if and only if the two independent sets are disjoint. Experiment result shows that the clique number of G is 2. Therefore, by Lemma 1, we have

Theorem 2 $\chi(KG^2(11,5))\geqslant 8$.

2 The upper bound for $\chi(KG^2(11,5))$

The graph coloring problem is a famous difficult combinatorial optimization NP-complete problem. There are many heuristic coloring algorithms to solve them, and the tabu search method is a popular one. Most of the recent heuristics for the graph coloring problem started from an infeasible k-coloring and tried to make the solution feasible through a sequence of color exchanges. In contrast, the approach in another reference[10], which was based on tabu search, considered feasible but partial solutions and tried to increase the size of the current partial solution.

The heuristic coloring algorithm based on tabu search[11] is used to color the graph $KG^2(11,5)$. Experiment result shows that the graph $KG^2(11,5)$ is 10-colorable, which is a record-breaking result. Thus we have.

Theorem 3 $\chi(KG^2(11,5))\leqslant 10$.

3 Discussion

To obtain the new lower bound or the exact value of $\chi(KG^2(11,5))$ needs a very large mount of computation. To reduce the computation, it is easy to see that we may obtain the equivalent independent sets and only consider the equivalent class. Here, we say two independent sets A and B are equivalent if there exists a permutation f on $\{1,2,\cdots,$

11} such that the elements of A can be transformed to those of B under f. For any independent set of $KG^2(11,5)$, we may suppose that there exists an equivalent independent set containing the element 1, 2,3,4,5. More detailed analyses in details will be helpful, but still not enough to compute the chromatic number of $KG^2(11,5)$ in a reasonable time.

An idea to try is as follows. Suppose V_1 is a maximum independent set in $KG^2(11,5)$. As we know $|V_1|=66$. Let H be $KG^2(11,5)-V_1$. So the order of H is $462-66=396$. If H is 8-colorable, then $\chi(KG^2(11,5))\leqslant 9$, otherwise $\chi(KG^2(11,5))\geqslant\chi(H)\geqslant 9$. We have done some computation, and found that we cannot determine if H is 8-colorable. Although we can do similar discussion on $\chi(KG^2(13,6))$, it seems far from reach.

The graph $KG^2(13,6)$ is a 49-regular graph with 1716 vertices. It seems challenging to determine the chromatic number of $KG^2(13,6)$, even its independent number. Using the heuristic coloring algorithm, we found that $KG^2(13,6)$ is 16-colorable indicating that $\chi(KG^2(13,6))\leqslant 16$. Note by Theorem 1 we have only $\chi(KG^2(13,6))\geqslant 8$, and $\chi(KG^2(13,6))\geqslant 9$ if $\alpha(KG^2(13,6))\leqslant 214$. In fact, $cl(KG^2(13,6))=8$. It seems difficult to compute a good upper bound for $\alpha(KG^2(13,6))$.

Searching for an induced subgraph H for which $|V(H)|/\alpha(H)$ is large, obtaining a lower bound for the chromatic number of a graph will be a easier way than dealing with the chromatic number directly. Note that computing upper bounds on chromatic numbers may be used in algorithms on clique numbers, which can be used in researching lower bounds for Ramsey numbers and their generalizations[11].

Let the subgraph of $KG^2(13,6)$ induced by the first to 118th vertices be G_0. By computing we found that $\alpha(G_0)=13$ and $\chi(G_0)\leqslant 10$. Since $118>117=13\times 9$, we obtain that $\chi(G_0)\geqslant 10$. So $10=\chi(G_0)\leqslant\chi(KG^2(13,6))\leqslant 16$.

In fact, we can also obtain $\chi(KG^2(11,5))\geqslant 8$ in this way.

It may be interesting to use $KG^2(2k+1,k)$ as benchmark of the Independent Set Problem.

References:

[1] Kneser M. Aufgabe 300[J]. Jahresber Deutsch Math-verein, 1955(58):27.

[2] Lovász L. Kneser's conjecture, chromatic number, and homotopy[J]. Journal of Combinatorial Theory: Series A, 1978, 25:319-324.

[3] Bárány I. A short proof of Kneser's conjecture[J]. Journal of Combinatorial Theory: Series A, 1978, 25: 325-326.

[4] Matoušek J. A combinatorial proof of Kneser's conjecture[J]. Combinatorica, 2004, 24(1):163-170.

[5] Ziegler G M. Generalized Kneser coloring theorems with combinatorial proofs[J]. Invent Math, 2002, 147: 671-691.

[6] Araujo G, Dumitrescu A, Hurtado F. et al. On the chromatic number of some geometric type Kneser graphs [J]. Computational Geometry, 2005, 32:59-69.

[7] Kim S J, Nakprasit K. On the chromatic number of the square of the Kneser graph $K(2k+1,k)$ [J]. Graphs Combin, 2004, 20:70-90.

[8] Chen J, Lih K, Wu J. Coloring the square of the Kneser graph $KG(2k+1,k)$ and the Schrijver graph $SG(2k+2,k)$ [J]. Discrete Applied Mathematics, 2009, 157: 170-176.

[9] Khodkar A, Leach D. The Chromatic Number of $K^2(9,4)$ is 11[J]. Journal of Combinatorial Mathematics and Combinatorial Computing, 2009, 70:217-220.

[10] Blochliger I, Zufferey N. A graph coloring heuristic using partial solutions and a reactive tabu scheme[J]. Computers & Operations Research, 2008, 35(2): 960-975.

[11] Liang M L, Yin C, Luo H P, et al. Upper and lower bounds for generalized multigraph Ramsey numbers based on Turán numbers[J]. Guangxi Sciences, 2011, 18(3):187-188.

(责任编辑:尹　闯)